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CONTENTS

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COMPLETENESS OF THE REAL NUMBERS

CASPER GOFFMAN

In memory of O. F. G. Schilling

Completeness of the real numbers in the sense of Dedekind (every nonempty set which has an upper bound has a least upper bound) means that the set of real numbers is complete as a totally ordered set. Completeness of the real numbers in the sense of Cantor (for every Cauchy sequence of real numbers there is a real number to which it converges) means the set of real numbers is complete as a metric space. The real numbers are also complete as a totally ordered group. This kind of completeness was used by Hilbert [7] in his work on the foundations of geometry, and will be called completeness in the sense of Archimedes. Our purpose is to discuss these three different notions of completeness and the relationship between them.

1. The definitions. A set S is *totally ordered* if there is an order relation $\leqq$ in S which satisfies:

(a) for each pair $x, y \in S$, either $x \leqq y$ or $y \leqq x$, and both $x \leqq y$ and $y \leqq x$ hold if and only if $x = y$, and

(b) if $x, y, z \in S$ and $x \leqq y$ and $y \leqq z$ then $x \leqq z$. If $x \leqq y$ and $x \neq y$ we write $x < y$; if $x \leqq y$ we also write $y \geqq x$, and if $x < y$ we also write $y > x$. A set $A \subset S$ is called a *lower segment in S* if $A \neq \emptyset$, $A \neq S$ and if $x \in A$, $y \leqq x$ implies $y \in A$. A totally ordered set S is said to be *complete* (in the Dedekind sense) if each lower segment in S has a least upper bound. The real numbers form a complete totally ordered set.

A set S together with a mapping $d : S \times S \to R$ is called a *metric space* if

(a) for each $x, y \in S$, $d(x, y) \geqq 0$, and $d(x, y) = 0$ if and only if $x = y$,

(b) for each $x, y \in S$, $d(x, y) = d(y, x)$, and

(c) for each $x, y, z \in S$, $d(x, y) + d(y, z) \geqq d(x, z)$.

A sequence $\{x_n\}$ in S is a *Cauchy sequence* if for each $\varepsilon > 0$ there is an N such that $m, n > N$ implies $d(x_m, x_n) < \varepsilon$; $\{x_n\}$ *converges* to x if for each $\varepsilon > 0$ there is an N such that $n > N$ implies $d(x, x_n) < \varepsilon$. A metric space S is *complete* (in the Cantor sense) if for each Cauchy sequence $\{x_n\}$ in S there is an $x \in S$ such that $\{x_n\}$ converges to x. The real numbers, with the metric $d(x, y) = |x - y|$, form a complete metric space.

A set S is called an *abelian totally ordered group* if

(a) S is a totally ordered set,

(b) S is an abelian group with operation $+$ and identity 0, and

(c) the two structures are compatible in the sense that if $x, y \in S$, $x \leqq y$, and $z \in S$ then $x + z \leqq y + z$.

A totally ordered group is said to be *archimedean* if for every pair $x, y \in S$, with $x, y > 0$, each member of the pair is less than some multiple of the other (a multiple of x is nx, i.e., $x + \cdots + x$ with n summands, for some n).

Remark 1. An archimedean totally ordered group is necessarily abelian. However, nonarchimedean groups need not be abelian (see [**8**]).

In an abelian totally ordered group S, two elements $x, y \in S$, with $x, y > 0$, are said to be *relatively archimedean* if each is less than some multiple of the other.

Let S and $T \supset S$ be abelian totally ordered groups, with S an ordered subgroup of T, i.e., the order relation and group operation in S are those induced by T. We say that T is an *archimedean extension* of S if for each $x \in T$, $x > 0$, there is a $y \in S$ such that x and y are relatively archimedean.

Remark 2. The terminology archimedean extension refers to the relationship between S and its extension T. If S is archimedean then so is each of its archimedean extensions. If S is nonarchimedean then it may have archimedean extensions which are also nonarchimedean.

The situation may be clarified by the following example:

Example 1. Let S be the set of ordered pairs (x, y) of integers. Order S lexicographically, i.e., $(x, y) < (u, v)$ if $x < u$ or if $x = u$ and $y < v$.

Let T be the set of ordered pairs of real numbers, also ordered lexicographically. With the usual group operation $(x, y) + (u, v) = (x + u, y + v)$, S and T are easily seen to be abelian totally ordered groups with S an ordered subgroup of T. Geometrically, T is the set of points in the plane and S is the set of lattice points with integer coordinates. A point comes before another if its x coordinate is smaller: if the x coordinates are the same, the point whose y coordinate is smaller comes first.

(a) *S is nonarchimedean.* For, $(0,0) < (0,1) < (1,0)$, and $n(0,1) = (0,n) < (1,0)$ for each $n = 1, 2, \cdots$.

(b) *T is an archimedean extension of S.* Let $(x, y) \in T$, $(x, y) > (0,0)$. Then either $x > 0$ or $x = 0$ and $y > 0$. If $x > 0$ then (x, y) and $(1,0)$ are relatively archimedean. If $x = 0$ and $y > 0$ then (x, y) and $(0,1)$ are relatively archimedean. But $(1,0)$ and $(0,1)$ are in S.

We also note that pictorially S is composed of $\aleph_0$ copies of the integers laid side by side in a double sequence and T is similarly composed of c copies of the reals.

An abelian totally ordered group is said to be *complete in the sense of Archimedes* if it has no proper archimedean extension.

THEOREM 1. *The real number system R is complete in the sense of Archimedes.*

Proof. The proof is rather easy. Let $H \supset R$ be an archimedean extension of R. Let $x \in H$, $x > 0$. We note that each positive real r is relatively archimedean with x since there is a $y \in R$, $y > 0$, which is relatively archimedean with x and r is relatively archimedean with y. In particular, $1/2^n$ is relatively archimedean with x. Accordingly, there is a positive integer j such that $(j-1)/2^n \leqq x \leqq j/2^n$. The intervals $I_n = [(j-1)/2^n, j/2^n]$ form a nested sequence $\{I_n\}$ of closed intervals in R with lengths converging to zero, so there is a unique real number $y \in I_n$, $n = 1, 2, \cdots$. Then, letting $|x - y|$ denote that one of $x - y$ or $y - x$ whch is $\geqq 0$, we have $|x - y| < 1/n$, for each n, so that $n|x - y| < 1$. On the other hand, since H is assumed to be an archimedean extension of R, $|x - y| > 0$ implies that there is an n for which $n|x - y| > 1$. This contradiction forces $|x - y| = 0$ so that $x = y$ and $H = R$. Thus, R is complete in the sense of Archimedes.

We note some examples.

(a) The real numbers are complete in all three senses.

(b) The integers are complete in the Dedekind sense and in the Cantor sense. They are not complete in the sense of Archimedes since the reals are a proper Archimedean extension.

(c) The rationals are not complete in any of the three senses.

2. Defects in the definitions. Consider the totally ordered group T of example 1. The members of T are ordered pairs of real numbers and T is ordered lexicographically. The set A in T defined by

$$A = [(x,y):(x,y) < (0,n) \text{ for some } n] = [(x,y):x \leqq 0]$$

is a lower segment. Suppose (u,v) is an upper bound of A. Then $u > 0$. So, $(u,v-1)$ is an upper bound of A. But $(u,v-1) < (u,v)$ so that A does not have a least upper bound.

Not only is this particular nonarchimedean group not complete in the Dedekind sense but we have the following general fact (e.g., [**1**]):

THEOREM 2. *If S is a nonarchimedean abelian totally ordered group, then S is not complete in the Dedekind sense.*

Proof. There are $x, y \in S$ with $0 < x < y$ and $nx < y$ for all $n = 1,2,\cdots$. The set

$$A = [z : z < nx \text{ for some } n = 1,2,\cdots]$$

is a lower segment in S. A has an upper bound; y is one. Suppose u is an upper bound of A. Then $u - x$ is also an upper bound. Suppose it is not. Then there is a $z \in A$ with $z > u - x$. But $z < nx$ for some nx. So $u - x < nx$ and $u + x < (n+2)x$ which places $u + x$ in A and violates the assumption that u is an upper bound of A. Accordingly, A does not have a least upper bound.

The fact that no nonarchimedean group can be complete in the Dedekind sense may be considered to be a defect in the definition. In section 3 we shall remedy this defect by making an appropriate modification of the definition of Dedekind completeness.

We first turn however to the notion of completeness in the Cantor sense. There will also be a defect here which we describe in some detail. We use some elementary facts about ordinal numbers. The *order type* of a totally ordered set S is the equivalence set to which S belongs, two totally ordered sets being equivalent if there is a one-one order preserving correspondence between them. A totally ordered set S is *well ordered* if each nonempty subset of S has a first element. The order type of a well ordered set is called an *ordinal number.* The smallest ordinal numbers are the finite ones. The first infinite ordinal number, the order type of the set of positive integers in their usual order is designated by ω_0. Its cardinal number is $\aleph_0$. There are many ordinal numbers whose cardinal number is $\aleph_0$. For example, the ordinal number $\omega_0 + 1$ of the well ordered set $2, 3, \cdots ; 1$ and the ordinal number $\omega_0 + \omega_0$ of the well ordered set $1, 3, 5, \cdots ; 2, 4, 6, \cdots$ both have cardinal number $\aleph_0$. However, ω_0 is the smallest ordinal number whose cardinal number is $\aleph_0$.

The set of all ordinal numbers whose cardinal numbers are finite or $\aleph_0$ forms a well ordered set. Its ordinal number is ω_1 and ω_1 is the smallest ordinal number whose cardinal number exceeds $\aleph_0$ (e.g., [**4**]).

(a) The ordinal number ω_0 has the following property: For any finite set $\alpha_1, \cdots, \alpha_n$ of ordinal numbers, each smaller than ω_0, there is an ordinal number $\alpha > \alpha_i$, $i = 1, \cdots, n$ with $\alpha < \omega_0$. That is to say, for any finite set of finite ordinal numbers there is a finite ordinal number greater than all of them.

(b) The ordinal number ω_1 has the following property: For any countably infinite set $\alpha_1, \alpha_2, \cdots$ of ordinal numbers smaller than ω_1 there is an $\alpha > \alpha_n$, $n = 1, 2, \cdots$, such that $\alpha < \omega_1$.

Now, let S be a well ordered set whose ordinal number is ω_1 and let G be the abelian totally ordered group of all real functions on S ordered lexicographically. This means that $x < y$ if $x(\alpha) < y(\alpha)$ for the smallest α for which they differ. It follows from the property of ω_1 given in (b) above that, given any sequence $x_1 > x_2 > \cdots > x_n > \cdots > 0$ in G, there is an $x > 0$ in G such that $x_n > x$ for all $n = 1, 2, \cdots$. Indeed, for each n, let α_n be the first ordinal number for which $x_n(\alpha_n) \neq 0$. Then $x_n(\alpha_n) > 0$. There is an $\alpha < \omega_1$ such that $\alpha > \alpha_n$, $n = 1, 2, \cdots$. Let x be defined by $x(\alpha) = 1$ and $x(\beta) = 0$ for each $\beta \neq \alpha$. Then $0 < x < x_n$, $n = 1, 2, \cdots$. It follows that there is no sequence of positive elements in G which converges to 0. In order to have a reasonable notion of Cantor completeness for abelian totally ordered groups, in general, it is accordingly necessary to consider transfinite sequences in contrast to ordinary sequences x_n, $n < \omega_0$.

For completeness in the sense of Archimedes there is no need to make any alterations in the definition.

3. Completeness in the sense of Dedekind. Since no abelian totally ordered group which is nonarchimedean can be complete in the Dedekind sense it is necessary to modify the definition in order to obtain a sufficiently broad theorem.

This is accomplished by allowing only certain lower segments. We define a lower segment A in an abelian totally ordered group S to be *dedekindean* if, for each $x \in S$, $x > 0$, there is a $y \in A$ such that $x + y \notin A$. We say two lower segments are equivalent if they differ by a single element. Then every equivalence class consists of one or two members. In the latter case, A has a least upper bound in S and we may either include it or exclude it. For convenience, we shall consider the representative which does not have a largest element.

We define operations for lower segments. If A and B are lower segments, their *sum* $A + B$ is defined by

$$A + B = [u \in S : u = x + y,\ x \in A,\ y \in B].$$

It is easy to see that $A + B$ is a lower segment and that if A and B are dedekindean then so is $A + B$.

For lower segments we define $A \leqq B$ if $A \subset B$.

The lower segment determined by 0 is designated as θ. Then $\theta = [x: x < 0]$.

For any lower segment A, a lower segment B is called an *inverse* of A if $A + B = \theta$. We now come to the crux of the matter about dedekindean lower segments.

THEOREM 3. *A lower segment A has an inverse if and only if it is dedekindean.*

Proof. (a) Suppose A is dedekindean. Let $B = [x: -x \notin A]$. We first show that B is a dedekindean lower segment. Let $x \in B$ and $y < x$. Since $-x \notin A$ and $-y > -x$ we have $-y \notin A$ so that $y \in B$. Next, let $x \in A$. Then $-x \notin B$ so that $B \neq S$. Moreover, if $x \notin A$ then $-x \in B$ so that $B \neq \varnothing$. Hence B is a lower segment. In order to show that B is dedekindean, let $x > 0$. There is $y \in A$ with $y + x \notin A$. Then $-y - x \in B$ and $-y = (-y - x) + x \notin B$.

We now show that $A + B = \theta$. Let $u < 0$. There is $y \in A$ with $-u + y \notin A$. Then $u - y \in B$ and $u = y + (u - y) \in A + B$. So if $u < 0$ then $u \in A + B$. On the other hand, if $x \in A$, $y \in B$ then $-y \notin A$ so that $x < -y$, whence $x + y < 0$. So if $u \in A + B$ then $u < 0$. This shows that $A + B = \theta$.

(b) Suppose A is non dedekindean. There is an $x > 0$ such that for each $y \in A$, $x + y \in A$. If $A + C = \theta$, there are $u \in A$, $v \in C$ such that $u + v = -x$. But $u + x \in A$. So $u + v + x = 0$ and $u + v + x = (u + x) + v \in A + C$. This means $A + C \neq \theta$.

Theorem 3 yields the fact that the dedekindean lower segments in S form an abelian totally ordered group S^* which is an extension of S (for each $x \in S$ the lower segment $A_x = [y: y < x]$ is a dedekindean lower segment). We shall call S^* the *d-completion* of S. An abelian totally ordered group is said to be *d-complete* if all of its dedekindean lower segments are of the above form A_x.

If S and T are totally ordered sets with $S \subset T$ then S is said to be *dense* in T if $x, y \in T$ with $x < y$ implies there is $z \in S$ with $x < z < y$. We note that if S is an abelian totally ordered group then S is dense in S^*. This is clear since if A and B are dedekindean lower segments in S with $A \subset B$, $A \neq B$, then there is an $x \in S$ such that $x \in B$ and $x \notin A$. We also note that if S' is an extension of S, with S dense in S', then there is a one-one correspondence between the dedekindean lower segments in S and those in S' obtained in the natural way. We leave the easy, somewhat messy, details to the reader.

Two interesting facts follow.

PROPOSITION 1. *The d-completion of S is d-complete.*

PROPOSITION 2. *S is dense in its d-completion S^*, and if T is any abelian totally ordered group such that $S \subset T$ and S is dense in T, then T is isomorphic with an ordered subgroup of S^*.*

Indeed, the property of Proposition 2 has been taken as the definition of *d*-completeness [9] and this may be the most natural form.

4. Completeness in the Cantor sense. This form of completeness may be defined in terms of uniform structures, nests of closed intervals, or transfinite sequences. We shall use transfinite sequences but note that in view of the equivalence between the Cantor and Dedekind methods the reader may skip this section if he so wishes.

An abelian totally ordered group S is said to be *discrete* if there is an $x \in S$, $x > 0$, such that $y > 0$ implies $y \geqq x$.

We note that each nondiscrete abelian totally ordered group S has a characteristic ordinal $\xi^* = \xi^*(S)$ in terms of which convergence is defined. This is established in the following way. We first obtain the fact that there is a well ordered transfinite sequence x_α, $\alpha < \xi$, of positive elements in S such that $\alpha < \beta < \xi$ implies $x_\beta < x_\alpha$ and for each $x > 0$ there is an $\alpha < \xi$ such that $x_\alpha < x$. We now consider the set of all such ordinals ξ and let ξ^* be the smallest one. This is the characteristic ordinal $\xi^* = \xi^*(G)$. It has a property like the one discussed in Section 2 for the special ordinals ω_0 and ω_1. See [**2**] for details. The property is as follows:

For each $\eta < \xi^*$ and ordinals $\xi_\alpha < \xi^*$, $\alpha < \eta$, there is a $\xi < \xi^*$ such that $\xi_\alpha < \xi$ for all $\alpha < \eta$.

We define ξ^* Cauchy sequences and ξ^* convergence. A sequence x_α, $\alpha < \xi^*$, is called ξ^* *Cauchy* if for each $u \in S$, $u > 0$, there is a $\xi < \xi^*$ such that $\xi < \alpha$, $\beta < \xi^*$ implies $|x_\alpha - x_\beta| < u$. A sequence x_α, $\alpha < \xi^*$, is said to be ξ^* *convergent* to x if for each $u \in S$, $u > 0$, there is a $\xi < \xi^*$ such that $\xi < \alpha < \xi^*$ implies $|x - x_\alpha| < u$. S is said to be *C-complete* if each ξ^* Cauchy sequence in S is ξ^* convergent.

Let S be a nondiscrete abelian totally ordered group with characteristic ordinal $\xi^* = \xi^*(S)$. We show that there is a one-one correspondence between equivalence classes of ξ^* Cauchy sequences and dedekindean lower segments in S, two ξ^* Cauchy sequences x_ξ, $\xi < \xi^*$, and y'_ξ, $\xi < \xi^*$, being equivalent if for each $u > 0$ there is an $\eta < \xi^*$ such that $\eta < \xi < \xi^*$ implies $|x_\xi - y_\xi| < u$.

For this purpose let $a_\xi > 0, \xi < \xi^*$, be ξ^* convergent to 0 and let A be a dedekindean lower segment in S. For each $\xi < \xi^*$ there is an $x_\xi \in A$ such that $x_\xi + a_\xi \notin A$. This transfinite sequence is associated with A. It is not hard to show:

(a) x_ξ, $\xi < \xi^*$, is a ξ^* Cauchy sequence.

(b) If x_ξ, $\xi < \xi^*$, and y_ξ, $\xi < \xi^*$, are associated in this way with distinct dedekindean lower segments, then x_ξ, $\xi < \xi^*$, and y_ξ, $\xi < \xi^*$, are not equivalent.

Conversely, let x_ξ, $\xi < \xi^*$, be a ξ^* Cauchy sequence. We associate a lower segment A with x_ξ, $\xi < \xi^*$, as follows: $x \in A$ if and only if there is a $\xi < \xi^*$ such that $x < x_\eta$ for all η satisfying $\xi < \eta < \xi^*$. It is not hard to show:

(a) A is a dedekindean lower segment.

(b) If A and B are associated with ξ^* Cauchy sequences x_ξ, $\xi < \xi^*$, and y_ξ, $\xi < \xi^*$, respectively, then $A = B$ if and only if the ξ^* sequences x_ξ, $\xi < \xi^*$, and y_ξ, $\xi < \xi^*$, are equivalent.

It follows from the above discussion that an abelian totally ordered group is d-complete if and only if it is C-complete, thereby obtaining the equivalence of the Cantor and Dedekind theories for arbitrary abelian totally ordered groups.

5. Hahn groups. We digress now to mention an important set of abelian totally ordered groups. These groups, introduced by H. Hahn [**5**], are groups of real functions on a totally ordered set. Let S be a totally ordered set and let H be the set of all real functions on S which are zero except on a subset of S which is well ordered with respect to the order relation in S. If x and y are in H it follows that $x + y$ is in H since the set on which $x + y$ is not zero will also be well ordered. It follows that H is an abelian group. We consider H to be ordered lexicographically, i.e., $x < y$ if for the smallest α for which $x(\alpha) \neq y(\alpha)$ we have $x(\alpha) < y(\alpha)$. Then H is an abelian totally ordered group.

We are now ready to state the Hahn representation theorem: *Every abelian totally ordered group is isomorphic with a subgroup of a Hahn group.* This theorem was proved by Hahn in 1907 and is modern in conception and method of proof. It is a pioneering effort in a type of reasoning which has prevailed especially in the thirties and forties. The proof of Hahn was written in a leisurely manner. For a sharp, precise, economical, altogether modern proof, the reader may consult the book, *Grundzüge der Mengenlehre*, written by Hausdorff in 1914 [**6**].

6. Relationship between *d*-completeness and *a*-completeness. Recall that an abelian totally ordered group is *a*-complete (complete in the sense of Archimedes) if it has no proper archimedean extension. We first prove the

THEOREM 4. *If S is a-complete, then it is d-complete.*

Proof. This follows from the fact, which we prove, that S^* is an archimedean extension of S. We may assume S nondiscrete. Let $x \in S^*$, $x > \theta$. Then there is a $y \in S^*$, $y > \theta$, with $2y \leqq x$. For, if $y < x$, $\theta < y$, then either $2y \leqq x$ or $2(x - y) \leqq x$. Now, suppose $2y \leqq x$. Since S is dense in S^* there is $z \in S$ such that $x - y < z < x$. It follows easily that $2z > x$ so that S^* is an archimedean extension of S.

The converse is false. We give two examples, the first of which is not deep.

(a) Let S be the group of integers. Then S is d-complete; but the group of reals is an archimedean extension of S so that S is not a-complete. Indeed, every discrete abelian totally ordered group is d-complete but not a-complete.

(b) Let S be the abelian totally ordered group of sequences of integers ordered lexicographically. Then $f < g$ if $f(n) < g(n)$, where n is the first positive integer for which $f(n) \neq g(n)$. If for each n we take $h_n(n) = 1$ and $h_n(m) = 0$ for $m \neq n$ then $\{h_n\}$ is a sequence of positive elements which converges to 0. It follows that a sequence $\{f_n\}$ in S is a Cauchy sequence if and only if for each N there is an n such that $r, s > n$ implies $f_r(k) = f_s(k)$ for each $k \leqq N$.

It is now an easy matter to see that S is d-complete. For let $\{f_n\}$ be a Cauchy sequence. Then for each k define $f(k) = f_r(k)$ where r is so large that $f_n(k) = f_r(k)$ whenever $n \geqq r$. We see that f_n converges to f.

On the other hand, S is not a-complete since the group T of sequences of real numbers, ordered lexicographically, is an archimedean extension of S.

In order to be able to state the theorem relating d-completeness with a-completeness, we need the notion of convex subgroup.

Let S be an abelian totally ordered group. A subgroup $I \subset S$ is called *convex* if $x \in I$, $x > 0$, $y \in S$, $0 < y < x$, implies $y \in I$. Then I is simply an interval in S centered at 0. Let I_x and I_y be cosets of I. If $I_x \neq I_y$ and $x < y$ then for each $u \in I_x$, $v \in I_y$ it easily follows that $u < v$. We accordingly have an order relation between the cosets of I and it follows that the quotient group S/I is an abelian totally ordered group.

THEOREM 5. *An abelian totally ordered group is a-complete if and only if, for each convex subgroup I of S, the quotient group S/I is nondiscrete and d-complete.*

The proof is not easy and is given in [**3**].

We are grateful to R. O. Davies for his many remarks leading to the improvement of the conceptual structure and presentation of this article.

References

1. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, 3rd ed., Macmillan, New York, 1965, p. 92.

2. L. W. Cohen and Casper Goffman, The topology of ordered abelian groups, Trans. Amer. Math. Soc., 67 (1949) 310–319.

3. L. W. Cohen and Casper Goffman, On completeness in the sense of Archimedes, Amer. J. Math., 72 (1950) 747–751.

4. Casper Goffman, Real Functions, Rinehart, New York, 1953, pp. 122–134.

5. H. Hahn, Über die nichtarchimedischen Grössensysteme, Sitz. der Kais. Akad. der Wiss., Vienna, Sect. IIa, 116 (1907) 601–653.

6. F. Hausdorff, Grundzüge der Mengenlehre, Chelsea, New York, 1917, pp. 194–207.

7. D. Hilbert, Grundlagen der Geometrie, Leipzig, 1903.

8. O. F. G. Schilling, The Theory of Valuations, Amer. Math. Soc., New York, 1950, pp. 6–7.

9. Dana Scott, On completing ordered fields, in Applications of Model Theory to Algebra, Analysis and Probability, ed. by W. A. J. Luxemburg, New York, 1969, pp. 274–278.

CARDAN'S FORMULAS AND BIQUADRATIC EQUATIONS

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1. Introduction. In 1956, Dr. Chao-Hui Yang showed me an interesting way to use a 3×3 cyclic matrix to recall Cardan's formulas for the roots of a cubic equation. Recently, by using a 4×4 cyclic matrix in a similar manner, I discovered analogous formulas for the roots of a biquadratic equation. All the details for a cubic are given in Section 2. My results for a biquadratic are presented in Theorem 1 of Section 4; they are related to other solution techniques in Sections 5 and 6.

As coefficient domain, *suppose F is a field of characteristic $\neq 2, 3$ with the property: for each element γ in F, $X^2 = \gamma$ has a solution in F and $X^3 = \gamma$ has a solution in F.* In particular, the quadratic formula is applicable, and each second-degree polynomial over F has a root in F; thus, F contains a principal cube root ω

of unity and a principal fourth root i of unity. For definiteness, F can be the field C of complex numbers.

2. Cardan's formulas. Starting with the cyclic matrix

$$\begin{bmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{bmatrix}, \tag{1}$$

we can easily remember to write the identity

$$X^3 + (-3YZ)X + (Y^3 + Z^3) = (X + Y + Z)(X + \omega Y + \omega^2 Z)(X + \omega^2 Y + \omega Z). \tag{2}$$

The left member of (2) equals the determinant D of (1). The right member of (2) follows via row operations on (1). Thus, by addition to the first row (X, Y, Z) of the second row (Z, X, Y) and the third row (Y, Z, X), we see $X + Y + Z$ is a factor of D. With $\omega^3 = 1$, addition to (X, Y, Z) of $\omega^2(Z, X, Y)$ plus $\omega(Y, Z, X)$ shows $X + \omega Y + \omega^2 Z$ is a factor of D; etc..

By replacing Y by $-Y$ and Z by $-Z$ in (2), we obtain

$$X^3 + (-3YZ)X + (-Y^3 - Z^3) = \prod_{s=0}^{2} (X - \omega^s Y - \omega^{2s} Z). \tag{3}$$

Suppose α and β are elements of F. To solve the cubic equation

$$X^3 + \alpha X + \beta = 0, \tag{4}$$

we seek a solution (y_0, z_0) of

$$-3YZ = \alpha \text{ and } -Y^3 - Z^3 = \beta. \tag{5}$$

There are two cases.

(i) Suppose $\alpha \neq 0$ or $\beta \neq 0$. Let t_0 be a nonzero solution in F of

$$T^2 + \beta T + \left(-\frac{\alpha}{3}\right)^3 = 0,$$

and let y_0 be a solution in F of $Y^3 = t_0$. With $y_0 \neq 0$, set $z_0 = -\alpha/3y_0$. Then, (y_0, z_0) is a solution of

$$YZ = -\frac{\alpha}{3} \text{ and } (Y^3)^2 + \left(-\frac{\alpha}{3}\right)^3 = -\beta Y^3.$$

Thus, with $y_0 \neq 0$, we see (y_0, z_0) is a solution of (5).

(ii) Suppose $\alpha = 0$ and $\beta = 0$. Then, we set $y_0 = 0$ and $z_0 = 0$.

We substitute y_0 for Y and z_0 for Z in (3) to obtain

$$X^3 + \alpha X + \beta = \prod_{s=0}^{2} (X - \omega^s y_0 - \omega^{2s} z_0).$$

Consequently, *equation* (4) *has three roots* x_1, x_2, x_3 *in* F *given by*

$$x_{s+1} = \omega^s y_0 + \omega^{2s} z_0, \text{ for } s = 0,1,2.$$

Given a cubic equation $\overline{X}^3 + \alpha_1 \overline{X}^2 + \alpha_2 \overline{X} + \alpha_3 = 0$ over F, the substitution $\overline{X} = X - (\alpha_1/3)$ reduces it to the form (4). Thus, *any cubic equation over* F *is solvable in* F.

3. The determinant of a cyclic matrix. We shall use the following modification of topics in [**2**].

LEMMA. *Suppose* K *is a field which contains a primitive nth root* ρ *of unity*; *let* $K[X_1, \cdots, X_n]$ *be a polynomial ring over* K *in* n *variables*; *and set*

$$A = \begin{bmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_n & X_1 & \cdots & X_{n-2} & X_{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ X_3 & X_4 & \cdots & X_1 & X_2 \\ X_2 & X_3 & \cdots & X_n & X_1 \end{bmatrix}.$$

Then, $\det A = f_0 f_1 \cdots f_{n-1}$, where

$$f_s = X_1 + \rho^s X_2 + \rho^{2s} X_3 + \cdots + \rho^{(n-1)s} X_n, \text{ for } s = 0,1,\cdots,n-1.$$

Proof. For $k = 1,2,\cdots,n$, let R_k denote the kth row of A. Set

$$R = R_1 + \sum_{k=2}^{n} \rho^{(n-k+1)s} R_k.$$

We find $R = (f_s, \rho^{(n-1)s} f_s,\ \rho^{(n-2)s} f_s, \cdots, \rho^s f_s)$. Let B denote the matrix obtained when the first row of A is replaced by R. We observe $\det B = \det A$ and f_s divides $\det A$ in $K[X_1, \cdots, X_n]$. The polynomial ring is factorial [**1**], and the elements $f_0, f_1, \cdots, f_{n-1}$ are irreducible. If j and k are integers with $0 \leqq j < k \leqq n-1$, then $\rho^j \neq \rho^k$ and each common divisor of f_j and f_k is a unit. Thus, the product $f_0 f_1 \cdots f_{n-1}$ divides $\det A$. We set

$$\det A = q f_0 f_1 \cdots f_{n-1}.$$

In terms of total degree, we find

$$n = \deg(\det A) = \deg q + \sum_{s=0}^{n-1} \deg f_s = \deg q + n$$

and $\deg q = 0$. Hence, q belongs to K. In $\det A$ and in $f_0 f_1 \cdots f_{n-1}$, the coefficient of X_1^n is 1. This yields $q = 1$ and completes the proof.

4. Formulas for the roots of a biquadratic equation. With $K = F, n = 4$, and $\rho = i$ (where $i^2 = -1$), the Lemma gives

$$\begin{vmatrix} X & U & V & W \\ W & X & U & V \\ V & W & X & U \\ U & V & W & X \end{vmatrix} = \prod_{s=0}^{3} (X + i^s U + i^{2s} V + i^{3s} W).$$

We expand the determinant and replace U, V, W by $-U, -V, -W$ to obtain

(6)
$$\begin{aligned} &X^4 + (-2V^2 - 4UW)X^2 + (-4U^2V - 4VW^2)X \\ &\quad + (-U^4 + V^4 - W^4 + 2U^2W^2 - 4UV^2W) \\ &= \prod_{s=0}^{3} (X - i^s U - i^{2s} V - i^{3s} W). \end{aligned}$$

This identity leads to the following result.

THEOREM 1. *Suppose a, b, c are elements of F. When $a = b = c = 0$, set $u_0 = v_0 = w_0 = 0$; otherwise, let v_0 be a nonzero solution in F of*

(7) $$(4V^2)^3 + 2a(4V^2)^2 + (a^2 - 4c)(4V^2) - b^2 = 0,$$

and let u_0, w_0 be elements of F which satisfy

(8) $$UW = -\frac{v_0^2}{2} - \frac{a}{4} \text{ and } U^2 + W^2 = -\frac{b}{4v_0}.$$

Then, the biquadratic equation

(9) $$X^4 + aX^2 + bX + c = 0$$

has four roots x_1, x_2, x_3, x_4 in F given by

(10) $$x_{s+1} = i^s u_0 + i^{2s} v_0 + i^{3s} w_0, \text{ for } s = 0, 1, 2, 3.$$

Proof. First, we verify (u_0, v_0, w_0) is a solution of

(11) $$4UW = -2V^2 - a,$$

(12) $$4V(U^2 + W^2) = -b,$$

and

(13) $$(U^2 + W^2)^2 + 8UWV^2 = \frac{a^2 - 4c}{4}.$$

For $a = b = c = 0$, this is clear. For $v_0 \neq 0$, we rewrite (7) as

$$4^2V^2\left(\frac{a^2-4c}{4}-2(-2V^2-a)V^2\right)=b^2; \tag{14}$$

then, (u_0,v_0,w_0) is a solution of (7), (8), (11), (12), (14), and

$$4^2V^2\left(\frac{a^2-4c}{4}-8UWV^2\right)=4^2V^2(U^2+W^2)^2;$$

hence, with $v_0\neq 0$, (u_0,v_0,w_0) is also a solution of (13).

We use (11) to eliminate a from (13); thus, (u_0,v_0,w_0) satisfies

$$-2V^2-4UW=a, \tag{15}$$

$$-4U^2V-4VW^2=b, \tag{16}$$

and

$$-U^4+V^4-W^4+2U^2W^2-4UV^2W=c. \tag{17}$$

We substitute (u_0,v_0,w_0) in (6), (15), (16), and (17) to obtain

$$X^4+aX^2+bX+c=\prod_{s=0}^{3}(X-i^su_0-i^{2s}v_0-i^{3s}w_0).$$

Consequently, (9) has four roots in F given by (10).

5. Further information. In Theorem 1, the element $4v_0^2$ is a root of

$$Y^3+2aY^2+(a^2-4c)Y-b^2=0. \tag{18}$$

We proceed to relate all three roots of (18) to Theorem 1.

PROPOSITION. *Suppose u_0,v_0,w_0 in F satisfy* (11), (12), *and* (13). *Set*

$$r_1=(1+i)u_0+(1-i)w_0,\quad r_2=2v_0,\quad r_3=(1-i)u_0+(1+i)w_0. \tag{19}$$

Then, $r_1r_2r_3=-b$ and (18) *has three roots in F given by r_1^2, r_2^2, and r_3^2.*

Proof. We use (19), (11), (13), and (12) to obtain

$$\begin{aligned}(Y-r_1^2)(Y-r_2^2)(Y-r_3^2)&=(Y-r_2^2)(Y^2-8u_0w_0Y+4(u_0^2+w_0^2)^2)\\&=Y^3+(-4v_0^2-8u_0w_0)Y^2+(4(u_0^2+w_0^2)^2+32u_0w_0v_0^2)Y\\&\quad-16v_0^2(u_0^2+w_0^2)^2\\&=Y^3+2aY^2+(a^2-4c)Y-b^2\end{aligned}$$

and

$$r_1r_2r_3=4v_0(u_0^2+w_0^2)=-b.$$

In [4], equation (18) was given as a cubic resolvent for equation (9), and a different solution procedure was established. Next, we derive the solution formulas of [4] from Theorem 1.

THEOREM 2. *Suppose elements* r_1, r_2, r_3 *of* F *satisfy*

$$(Y - r_1^2)(Y - r_2^2)(Y - r_3^2) = Y^3 + 2aY^2 + (a^2 - 4c)Y - b^2 \tag{20}$$

and

$$r_1 r_2 r_3 = -b. \tag{21}$$

Then, equation (9) *has four roots* x_1, x_2, x_3, x_4 *in* F *given by*

$$\begin{aligned} x_1 &= (+r_1 + r_2 + r_3)/2, \\ x_2 &= (+r_1 - r_2 - r_3)/2, \\ x_3 &= (-r_1 + r_2 - r_3)/2, \\ x_4 &= (-r_1 - r_2 + r_3)/2. \end{aligned} \tag{22}$$

Proof. We define u_0, v_0, w_0 in F through

$$4u_0 = (1 - i)r_1 + (1 + i)r_3,\ 2v_0 = r_2,\ 4w_0 = (1 + i)r_1 + (1 - i)r_3. \tag{23}$$

Using (23), (20), and (21), we find

$$\begin{aligned} 4u_0 w_0 + 2v_0^2 &= \frac{r_1^2 + r_2^2 + r_3^2}{2} = -a, \\ 4v_0(u_0^2 + w_0^2) &= r_1 r_2 r_3 = -b, \text{ and} \\ (u_0^2 + w_0^2)^2 + 8u_0 w_0 v_0^2 &= \frac{r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2}{4} = \frac{a^2 - 4c}{4}. \end{aligned}$$

Thus, (u_0, v_0, w_0) is a solution of (11), (12), and (13). From the proof of Theorem 1, the roots of (9) are given by (10). For $s = 0, 1, 2, 3$, we use (10) and (23) to obtain (22).

6. Several observations. Given λ, μ in F, we specify a solution of

$$UW = \lambda \text{ and } U^2 + W^2 = \mu.$$

For $\lambda = \mu = 0$, set $u_0 = w_0 = 0$; otherwise, let t_0 be a nonzero solution in F of $T^2 - \mu T + \lambda^2 = 0$, let u_0 satisfy $U^2 = t_0$, and set $w_0 = \lambda / u_0$. In this way, (8) can be satisfied.

When $b = 0$ in equation (9), the conditions

$$V = 0,\ UW = -\frac{a}{4},\text{ and } (U^2 + W^2)^2 = \frac{a^2 - 4c}{4}$$

also specify a solution of (11), (12), and (13) as well as a solution procedure (10) for (9). We can take $\lambda = -a/4$ and μ so $\mu^2 = (a^2 - 4c)/4$. Of course, with $b = 0$, equation (9) can be solved directly as a quadratic in X^2.

Let S_1 be the set of arrangements (first, second, third, fourth) for the four roots of (9); let S_2 be the set of solutions of (11), (12), and (13); and, let S_3 be the set of triples (r_1, r_2, r_3) which satisfy (20) and (21). *There exists a bijection of* S_2 *onto* S_1. Namely, the mapping from S_2 to S_1 given by (10) is clearly injective. To prove it surjective, suppose (x_1, x_2, x_3, x_4) is an element of S_1; then, $x_1 + x_2 + x_3 + x_4 = 0$; by solving linear equations, we find unique elements u_0, v_0, w_0 in F which satisfy (10); by (6), (u_0, v_0, w_0) is a solution of (15), (16), and (17); hence, (u_0, v_0, w_0) is an element of S_2; etc. Similarly, *there exists a bijection of* S_3 *onto* S_1. Directly from (19) and (23), *the sets* S_2 *and* S_3 *are in one-to-one correspondence.*

Another procedure to deduce a cubic resolvent for (9) is given in [3]. Based upon Galois theory, the solution formulas of [3] are analogous to (18), (20), (21), and (22). The differences of notation are due to the changes necessitated when Y is replaced in (18) by $-Y$.

Given a biquadratic equation $\overline{X}^4 + a_1\overline{X}^3 + a_2\overline{X}^2 + a_3\overline{X} + a_4 = 0$ over F, the substitution $\overline{X} = X - (a_1/4)$ reduces it to the form (9). Thus, any biquadratic equation over F is solvable in F.

7. An example. In the equation

$$X^4 - 2\beta^2X^2 - 4\alpha^2\beta X + (\beta^4 - \alpha^4) = 0, \tag{24}$$

we suppose α and β are nonzero elements of F. With

$$a = -2\beta^2,\ b = -4\alpha^2\beta,\ c = \beta^4 - \alpha^4, \text{ and } (a^2 - 4c)(-2a) - b^2 = 0,$$

the corresponding cubic resolvent (18) has $-2a$ as a root.

To solve (24) by Theorem 1, we take

$$4v_0^2 = -2a = 4\beta^2,\ v_0 = \beta,\ UW = 0,\ U^2 + W^2 = \alpha^2,\ u_0 = \alpha, \text{ and } w_0 = 0.$$

Thus, by (10), the roots of (24) are

$$x_{s+1} = i^s\alpha + i^{2s}\beta, \text{ for } s = 0, 1, 2, 3. \tag{25}$$

In this situation, Theorem 2 is less direct. First, all three roots

$$2i\alpha^2,\ 4\beta^2,\ -2i\alpha^2$$

of (18) are required. Then, elements r_1, r_2, r_3 of F are needed to satisfy

$$r_1^2 = 2i\alpha^2,\ r_2^2 = 4\beta^2,\ r_3^2 = -2i\alpha^2, \text{ and } r_1r_2r_3 = 4\alpha^2\beta;$$

one choice is $r_1 = (1 + i)\alpha$, $r_2 = 2\beta$, and $r_3 = (1 - i)\alpha$. At this point, we can use (22) to obtain (25).

References

1. S. Lang, Algebra, Addison-Wesley, Reading, Massachusetts, 1965, p. 127.
2. T. Muir, A Treatise on the Theory of Determinants, Dover, New York, 1960, pp. 442–445.
3. B. L. van der Waerden, Modern Algebra, vol. 1, Ungar, New York, 1953, pp. 180–182.
4. H. Weber, Lehrbuch der Algebra, vol. 1, Chelsea, New York, 1963, pp. 135–136.

CONSTELLATION MORLEY

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1. Introduction. Morley's theorem is that the adjacent internal trisectors of the angles of a triangle intersect forming an equilateral triangle. This triangle is now called Morley's triangle. Let us name it **Morley's triangle *i* (Morley Δ*i*)** as its vertices lie within the mother triangle. A proof of this theorem has appeared in this MAGAZINE by G. L. Neidhardt and V. Milenkovic [1]. Analogously, W. R. Spickerman [2] has shown that the adjacent external trisectors of the angles of a triangle also intersect to form an equilateral triangle. Let us name it **Morley's triangle *e* (Morley Δ*e*)** as its vertices are outside the mother triangle.

In this paper we shall consider some results associated with a triangle, its Morley Δ*i* and Morley Δ*e*. We consider some relations between the trisectors (internal and external) of the angles of a triangle, some inequalities, several interesting concurrent theorems, a remarkable result that Morley Δ*e* is homothetic to Morley Δ*i*, and a converse of Morley's theorem. *Proofs are omitted with the hope that supplying them would be an interesting undergraduate research project.*

2. Preliminary results. Throughout this paper, ΔPQR (Figure 1) represents Morley Δ*i*, $\Delta P'Q'R'$ represents Morley Δ*e* of a ΔABC. $B\hat{A}C = 3\alpha$, $A\hat{B}C = 3\beta$ and $B\hat{C}A = 3\gamma$, $a = \overline{BC}$, $b = \overline{CA}$ and $c = \overline{AB}$ are the elements of ΔABC; $m = \overline{PQ} = \overline{QR} = \overline{RP}$; $m' = \overline{P'Q'} = \overline{Q'R'} = \overline{R'P'}$; D, the circumdiameter of ΔABC. Our notation is the same as [1] but differs from [2]. For the sake of brevity, let us recall some results from [1] and [2].

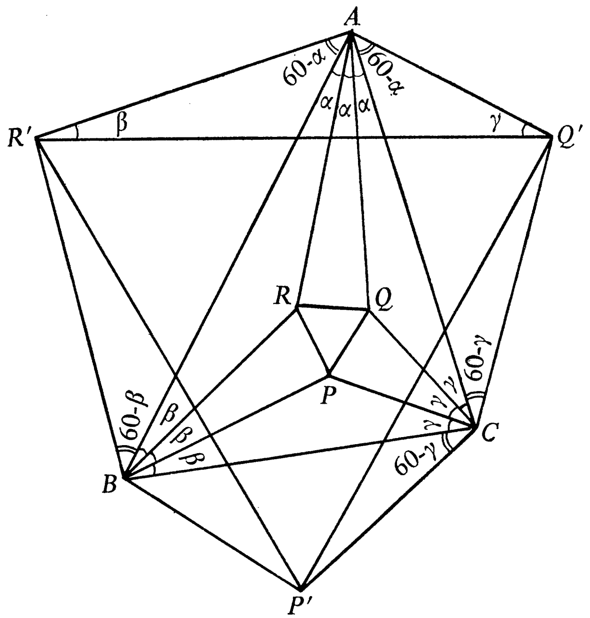

FIG. 1.

$$
(2.1)\quad \begin{cases}
A\hat{R}Q = C\hat{P}Q = 60+\beta; \quad A\hat{Q}R = B\hat{P}R = 60+\gamma; \\
B\hat{R}P = C\hat{Q}P = 60+\alpha; \\
\overline{PC} = 4D \sin\alpha \sin\beta \sin\overline{60+\alpha}; \\
\overline{QC} = 4D \sin\alpha \sin\beta \sin\overline{60+\beta}; \\
\overline{QA} = 4D \sin\beta \sin\gamma \sin\overline{60+\beta}; \\
\overline{RA} = 4D \sin\beta \sin\gamma \sin\overline{60+\gamma}; \\
\overline{RB} = 4D \sin\gamma \sin\alpha \sin\overline{60+\gamma}; \\
\overline{PB} = 4D \sin\gamma \sin\alpha \sin\overline{60+\alpha}; \\
m = 4D \sin\alpha \sin\beta \sin\gamma.
\end{cases}
$$

$$
(2.2)\quad \begin{cases}
A\hat{R}'Q' = C\hat{P}'Q' = \beta; \quad A\hat{Q}'R' = B\hat{P}'R' = \gamma; \\
B\hat{R}'P' = C\hat{Q}'P' = \alpha. \\
\overline{P'C} = 4D \sin\overline{60-\alpha} \sin\overline{60-\beta} \sin\alpha; \\
\overline{Q'C} = 4D \sin\overline{60-\alpha} \sin\overline{60-\beta} \sin\beta; \\
\overline{Q'A} = 4D \sin\overline{60-\beta} \sin\overline{60-\gamma} \sin\beta; \\
\overline{R'A} = 4D \sin\overline{60-\beta} \sin\overline{60-\gamma} \sin\gamma; \\
\overline{R'B} = 4D \sin\overline{60-\gamma} \sin\overline{60-\alpha} \sin\gamma; \\
\overline{P'B} = 4D \sin\overline{60-\gamma} \sin\overline{60-\alpha} \sin\alpha; \\
m' = 4D \sin\overline{60-\alpha} \sin\overline{60-\beta} \sin\overline{60-\gamma}.
\end{cases}
$$

3. Equality property of trisectors. From (2.1) it is immediate that

$$(3.1)\qquad \overline{BP}\cdot\overline{CQ}\cdot\overline{AR} = \overline{BR}\cdot\overline{CP}\cdot\overline{AQ}.$$

In fact it can be shown that each product is equal to $(m/m')abc$.

From (2.2), we obtain

$$(3.2)\qquad \overline{BP'}\cdot\overline{CQ'}\cdot\overline{AR'} = \overline{BR'}\cdot\overline{CP'}\cdot\overline{AQ'} = mm'^2.$$

Then (3.1) and (3.2) yield:

$$(3.3)\qquad \overline{BP}\cdot\overline{BP'}\cdot\overline{CQ}\cdot\overline{CQ'}\cdot\overline{AR}\cdot\overline{AR'} = \overline{BR}\cdot\overline{BR'}\cdot\overline{CP}\cdot\overline{CP'}\cdot\overline{AQ}\cdot\overline{AQ'}$$
$$= m^2m'abc,$$

a result that expresses the product of the distances of the vertices of a ΔABC from those of its Morley triangles, taken in order, purely in terms of the side lengths of these triangles.

4. Trigonometrical inequalities. The expressions for m and m' themselves give rise to two trigonometrical inequalities. Using $\alpha + \beta + \gamma = 60$ and changing the product of sines to sums, we find,

$$\frac{\sqrt{3}}{2} \leqq \Sigma \sin(60 - 2\alpha) \leqq 3 \sin 20 \quad \text{and} \tag{4.1}$$

$$\frac{\sqrt{3}}{2} \leqq \Sigma \sin 2\alpha \leqq 3 \sin 40. \tag{4.2}$$

In terms of the angles A, B and C, these are,

$$\frac{\sqrt{3}}{2} \leqq \Sigma \sin\left(60 - \frac{2A}{3}\right) \leqq 3 \sin 20 \quad \text{and,} \tag{4.1*}$$

$$\frac{\sqrt{3}}{2} \leqq \Sigma \sin \frac{2A}{3} \leqq 3 \sin 40. \tag{4.2*}$$

5. Concurrence theorems. The equalities (3.1) and (3.2) remind us of Ceva's theorem concerning the concurrent lines through the vertices of a triangle and prompt us to look for such lines. We would like to remark that these results are especially interesting in that not only does a triangle have a set of concurrent lines such as the angle bisectors but these also generate concurrent lines in the other two triangles.

For example consider the internal bisectors of the angles of ΔABC meeting QR, RP, PQ in F_1, F_2, F_3 respectively. Then AF_1, BF_2, CF_3 are also bisectors of the angles QAR, RBP and PCQ; and $(\overline{RF_1}/\overline{F_1Q}) = (\overline{AR}/\overline{AQ}) = (\sin \overline{60 + \gamma}/\sin \overline{60 + \beta})$, etc. Similar expressions and the converse of Ceva's theorem show that

$$PF_1,\ QF_2 \text{ and } RF_3 \tag{5.1}$$

are concurrent.

If

$$F'_1,\ F'_2 \text{ and } F'_3 \tag{5.1'}$$

are the points where the internal bisectors of the angles of ΔABC meet $Q'R'$, $R'P'$ and $P'Q'$ respectively, then $P'F'_1$, $Q'F'_2$ and $R'F'_3$ are concurrent.

(5.1*, 5.1'*). If we replace "*internal bisectors*" in (5.1) and (5.1') by the corresponding *external bisectors*, the results are still true.

(5.2, 5.2*). Suppose the internal (external) bisectors of the angles P, Q, R of ΔPQR meet the sides BC, CA, AB of ΔABC in the points G_1, G_2, G_3 respectively. Then the lines AG_1, BG_2, and CG_3 are concurrent.

(5.2′, 5.2′*). Suppose the internal (external) bisectors of the angles P', Q' and R' of $\Delta P'Q'R'$ meet the sides BC, CA, AB of ΔABC in G_1', G_2' and G_3', then AG_1' BG_2' and CG_3' are concurrent.

(5.3, 5.3*). Let the bisectors, internal or external, of the angles BPC, CQA and ARB meet the sides BC, CA, AB of ΔABC in H_1, H_2, H_3 respectively. Then AH_1, BH_2 and CH_3 are concurrent.

(5.3′, 5.3′*). Let the bisectors, internal or external, of the angles $BP'C$, $CQ'A$, and $AR'B$ meet the sides BC, CA and AB of ΔABC in H_1', H_2' and H_3' respectively. Then AH_1', BH_2', and CH_3' are concurrent.

(5.4). Let the perpendiculars from A, B, and C to the sides QR, RP and PQ meet them respectively in K_1, K_2 and K_3. Then PK_1, QK_2, and RK_3 are concurrent.

(5.4′). Let the perpendiculars from A, B, and C to the sides $Q'R'$, $R'P'$ and $P'Q'$ meet them respectively in K_1', K_2' and K_3'. Then PK_1', QK_2' and RK_3' are concurrent.

(5.5). Let the perpendiculars from P, Q and R to the sides BC, CA and AB meet them in L_1, L_2 and L_3 respectively. Then AL_1, BL_2 and CL_3 are concurrent.

(5.5′). Let the perpendiculars from P', Q' and R' to the sides BC, CA and AB meet them in L'_1, L'_2, and L'_3 respectively. Then AL_1', BL_2' and CL_3' are concurrent.

6. The corresponding sides of Morley Δs *i* and *e* are parallel. Let BR and CQ meet at P_1 (Figure 2). J. C. Burns [3] has shown that $\overline{P_1Q} = \overline{P_1R}$ and that PP_1 bisects $Q\hat{P}R$ and $Q\hat{P}_1R$. Let the bisectors of the angles $Q\hat{P}R$ and $Q'P'R'$ meet BC in G and G' respectively. It can be proved that,

$$PG \parallel P'G', \tag{6.1}$$

and hence that,

$$QR \parallel Q'R', \tag{6.2}$$

etc., and thus the remarkable conclusion. We note that,

$$PP',\ QQ' \text{ and } RR' \tag{6.3}$$

are concurrent.

7. Another concurrence theorem. Once we know that $QR \parallel Q'R'$, it is easy to prove that.

$$PQ,\ BR \text{ and } Q'R' \tag{7.1}$$

are concurrent. Let L_1 be the point of concurrence. Five similar results give rise to the points L_2, M_1, M_2 and N_1, N_2.

(7.11). The six points L_1, L_2, M_1, M_2, N_1, N_2 form a set of important points of the constellation.

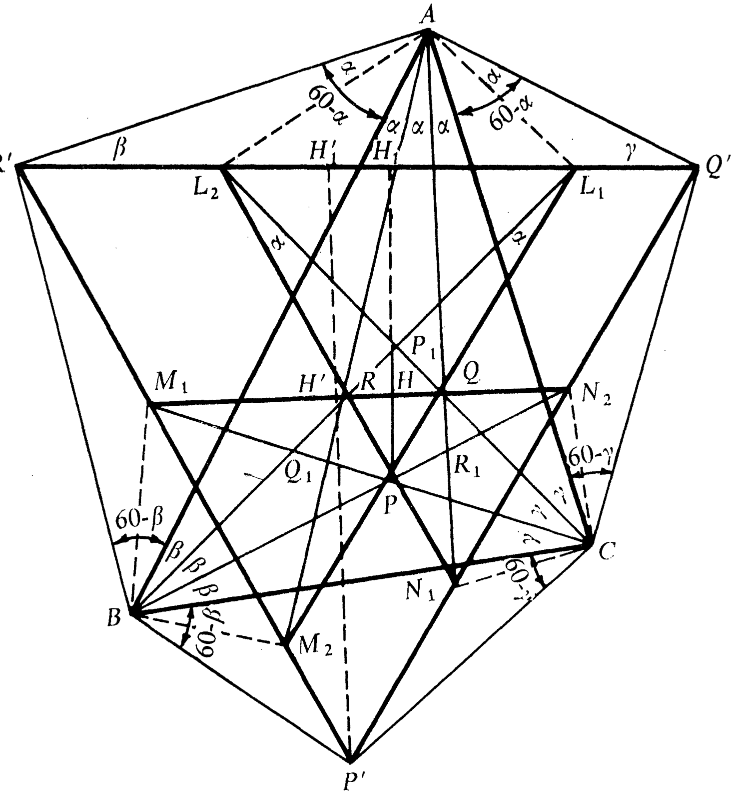

FIG. 2.

8. More distance relations. We can derive that

$$\overline{AL_1} = 4D \sin\beta \sin\gamma \sin\overline{60-\gamma}; \tag{8.1}$$

$$\overline{AL_2} = 4D \sin\beta \sin\gamma \sin\overline{60-\beta};$$

$$\overline{BL_1} = 4D \sin\overline{60+\alpha} \sin\overline{60-\gamma} \sin\gamma;$$

and similar expressions. From these follow,

$$\overline{AL_1} \cdot \overline{BM_1} \cdot \overline{CN_1} = \overline{AL_2} \cdot \overline{BM_2} \cdot \overline{CN_2} = m^2 m' \tag{8.11}$$

$$\overline{BL_1} \cdot \overline{CM_1} \cdot \overline{AN_1} = \overline{CL_2} \cdot \overline{AM_2} \cdot \overline{BN_2} = abc. \tag{8.12}$$

$$\overline{AL_1} \cdot \overline{BL_1} \cdot \overline{BM_1} \cdot \overline{CM_1} \cdot \overline{CN_1} \cdot \overline{AN_1} = \overline{AL_2} \cdot \overline{CL_2} \cdot \overline{BM_2} \cdot \overline{AM_2} \tag{8.13}$$

$$\cdot \overline{CN_2} \cdot \overline{AN_2} = m^2 m' abc = \overline{BP} \cdot \overline{BP'} \cdot \overline{CQ} \cdot \overline{CQ'} \cdot \overline{AR} \cdot \overline{AR'}$$

$$= \overline{BR} \cdot \overline{BR'} \cdot \overline{CP} \cdot \overline{CP'} \cdot \overline{AQ} \cdot \overline{AQ'}.$$

(8.2) $$\overline{L_1L_2} = 4D \sin \overline{60 + \alpha} \sin\beta \sin\gamma$$

gives the side length of the equilateral ΔPL_1L_2. Similar expressions can be derived giving the side lengths of the equilateral Δs QM_1M_2 and RN_1N_2.

(8.21) $$\overline{L_1L_2} \cdot \overline{M_1M_2} \cdot \overline{N_1N_2} = \frac{m}{m'} abc = \overline{BP} \cdot \overline{CQ} \cdot \overline{AR}$$

$$= \overline{BR} \cdot \overline{CP} \cdot \overline{AQ}.$$

Since $\overline{QR} \parallel \overline{Q'R'}$, we may think of finding the distance $\overline{HH_1}$ $(= l_1)$ separating them. In fact,

(8.3) $$l_1 = 2\sqrt{3}\, D \sin \overline{60 - \alpha} \sin\beta \sin\gamma.$$

If l_2 and l_3 are the distances between $\overline{RP}$ and $\overline{R'P'}$, and between $\overline{PQ}$ and $\overline{P'Q'}$, then,

(8.31) $$l_1 l_2 l_3 = \frac{3\sqrt{3}}{8} m^2 m' = \frac{3\sqrt{3}}{8} \overline{AL_1} \cdot \overline{BM_1} \cdot \overline{CN_1} = \frac{3\sqrt{3}}{8} \overline{AL_2} \cdot \overline{BM_2} \cdot \overline{CN_2}.$$

Also,

(8.4) $$\overline{Q'L_1} \cdot \overline{R'M_1} \cdot \overline{P'N_1} = \overline{AL_1} \cdot \overline{BM_1} \cdot \overline{CN_1}.$$

Let the distance between P' and $\overline{QR}$, i.e., $\overline{P'H'}$ be d_1. Then,

(8.5) $$d_1 = 3D \sin\alpha \sin \overline{60 - \alpha}.$$

If we define d_2 and d_3 similarly, then

(8.51) $$d_1 \cdot d_2 \cdot d_3 = \frac{27}{16} Dmm'.$$

Further,

$$d_1 = \frac{3D}{2}[\cos \overline{60 - 2\alpha} - \cos 60].$$

$$\therefore \Sigma d_1 = \frac{3D}{2} \Sigma(\cos \overline{60 - 2\alpha} - \cos 60) = \frac{3D}{2} [4 \cos\alpha \cos\beta \cos\gamma - 2],$$

which implies

(8.52) $$\frac{1}{2} \leqq \cos\frac{A}{3} \cdot \cos\frac{B}{3} \cdot \cos\frac{C}{3} \leqq \frac{1}{8} + \frac{3}{4}\cos 20.$$

Lastly, we may think of calculating $\overline{H_1H_1'}$, the distance between the corresponding bisectors PG and $P'G'$. Then,

(8.6) $$\overline{H_1H_1'} = \sqrt{3}\, D \sin\alpha \sin \overline{\beta - \gamma}.$$

9. Some observations.

(9.1). Through each of the vertices A, B, and C there are three triangles similar to each other corresponding to the trisectors of these angles of ΔABC with angles α, $60+\beta$ and $60+\gamma$.

(9.2). There is a set of seven similar triangles with angles α, β and $120+\gamma$, and two more similar sets.

(9.3). There are three cyclic pentagons.

(9.4). The configuration is a hexagon, with three pairs of opposite angles, each pair adding up to 240°.

(9.5). Morley Δe is neatly partitioned into seven nonoverlapping regions — three equiangular parallelograms of which any two have a pair of equal sides, three equiangular isosceles trapeziums and Morley Δi. Of course, this should be so.

10. A nine point star. In (7.11) we mentioned that the six points L_1, L_2, M_1, M_2 and N_1, N_2 form a set of important points of the Constellation. We now see how this can be used to obtain the following converse of Morley's Theorem.

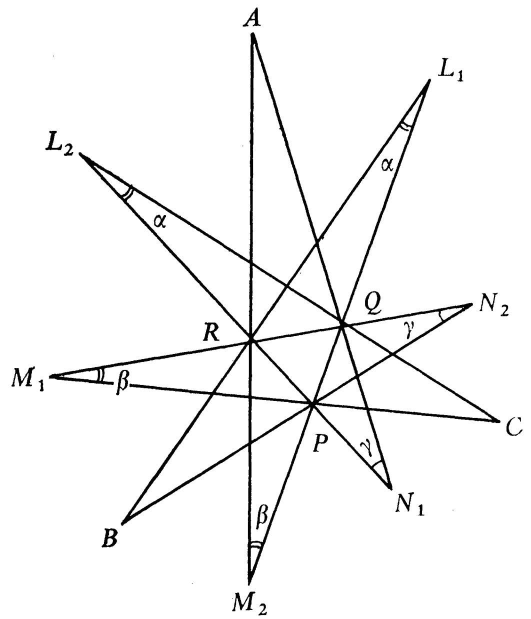

FIG. 3.

Let A, L_1, L_2; B, M_1, M_2; and C, N_1, N_2 be a nine points tar (Figure 3) with $Q\hat{L}_1R = Q\hat{L}_2R = \alpha$; $R\hat{M}_1P = R\hat{M}_2P = \beta$ and $P\hat{N}_1Q = P\hat{N}_2Q = \gamma$, giving rise to the equilateral triangle PQR. *With the assumption that* $\alpha+\beta+\gamma = 60$, *or equivalently*, $Q\hat{A}R = \alpha$, *we can prove that*

10.1. ***ABC* is the triangle for which ΔPQR is Morley Δi.**

We can further prove that

10.2. **L_1L_2, N_1N_2, an external trisector of $B\hat{A}C$ and an external trisector of $A\hat{C}B$ are concurrent and that point is a vertex of Morley Δe.** (See Figure 4.)

It would be interesting to know whether the assumption that $\alpha + \beta + \gamma = 60$ can be dropped.

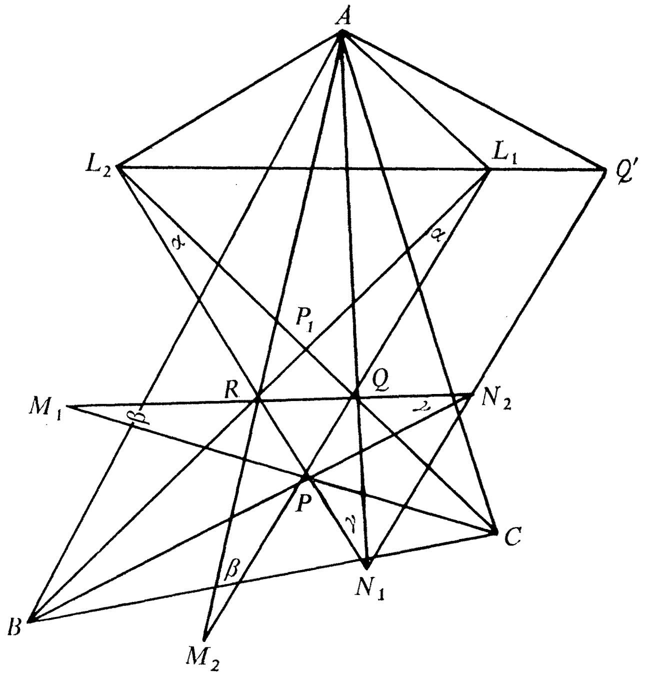

FIG. 4.

11. Conclusion. The Constellation reveals more interesting points if we have more patience to observe. It is truly a fine example of a Euclidean configuration with several beautiful properties concealed. We conclude this article with a remark that there should be many more and it is for the interested one to mine. Some interesting questions that are evident are simpler expressions for $\overline{PP'}$, $\overline{PQ'}$, etc., and the discovery of interesting relationships they may have.

Acknowledgement. The author wishes to thank the referee whose suggestions led to a better presentation.

References

1. G. L. Neidhardt and V. Milenkovic, Morley's triangle, this MAGAZINE, 42 (1969) 87–88.
2. W. R. Spickerman, An extension of Morley's theorem, this MAGAZINE, 44 (1971) 191–192.
3. J. C. Burns, Morley's triangle, this MAGAZINE, 43 (1970) 210–211.

THE GAME OF NIM—A HEURISTIC APPROACH

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The game of Nim is played by two persons. Out of a finite number of indistinguishable counters an arbitrary number of piles, each containing an arbitrary number of counters, are formed. The players alternate in making moves. In any move, the player must remove at least one counter from a pile of his choice. He may remove an entire pile. He cannot remove counters from different piles in the same move. The player who removes the last counter (by itself or together with others) wins the game.

In the mathematical literature, the game seems to have been described first by Bouton in a 1902 issue of the Annals of Mathematics. He also described the winning strategy of the game and proved its correctness. Since then, many papers have been published about the game and its strategy. The book, *An Introduction to the Theory of Numbers*, by Hardy and Wright (1954, pages 117–120) contains a solution of the game. To my knowledge, however, neither this book nor any of these papers describes how the solution of the strategy-problem was found. Yet, from the pedagogical point of view, how the solution of a problem is obtained is of crucial importance. This is especially true in the case of Nim, the strategy of which may seem to be rather far-fetched.

Some time in 1963, during a conversation with mathematician friends, the strategy of Nim came up, especially the question "how to solve it." When I remarked that I had solved it, the editor of the Hungarian High School Mathematics Magazine, suggested that publication of the solution process could be of pedagogical interest. The result of this suggestion was a paper published in that magazine in 1964. In it was given an outline of the chain of reasoning that led to the solution. The present paper is a revised translation of the original.

First some preliminary remarks: As Hardy and Wright put it, "The game has a precise mathematical theory, and one or the other player can always enforce a win." Indeed, there is a theorem of von Neumann according to which games with perfect information (like Nim) have a nonprobabilistic winning strategy. However, we don't use this theorem; we just *assume* the existence of a strategy and show how to guess its properties from actual playing experience. Our objective is to show how the winning strategy emerges from experimental observations.

As to terminology, the set of the numbers of counters in each pile after any particular move is called the "position" of the game at that point. We characterize a position by putting the number of counters in each pile in parentheses in a nondecreasing order. For example, (2, 4, 4, 7, 9) is a five-pile position. We call positions according to the number of their piles singletons, pairs, triplets, etc.

If both players know the assumed winning strategy then it is obvious that the outcome of the game is determined by its initial position. Therefore, there must be two kinds of positions: if the player to make the first move can win against an informed opponent the initial position is a winning one (w-position); if an informed player is to move and cannot force a win, the position is losing (l-position).

Experience in playing shows that l-positions represent a minority. Therefore, it is reasonable to look for definitive characteristics of an l-position rather than that of a w-position. Indeed, it would be a big step toward finding the winning strategy if one were able to recognize l-positions.

If there is a winning strategy the following conditions concerning the two kinds of positions are obvious:

1. An l-position is necessarily changed into a w-position by any single move.

2. It must be possible to change any w-position into an l-position by a single move.

3. General characterization of l-positions must include the simplest l-position (1,1).

Let us look now at some simple positions. Any singleton is a w-position no matter how many counters it contains. The case of a pair also admits a simple solution: (a, b) is an l-position if and only if $a = b$. In the position (a, a) any move would make the piles unequal, and a countermove could always restore equality.

If a triplet is an l-position, the three numbers must be different. Indeed, if the triplet is (a, b, b), removal of "a" would change it into the l-position (b, b).

The simplest triplet with unequal numbers is (1, 2, 3). Six moves are possible, resulting in the positions (0, 2, 3), (1, 1, 3), (1, 0, 3), (1, 2, 2), (1, 2, 1), (1, 2, 0). All these can be converted by a single move into obvious l-positions. Thus (1, 2, 3) is an l-position. Keeping the smallest number in the triplet 1, the next smallest triplet is (1, 2, 4), and the following one is (1, 3, 4). Both can be changed by a single move into (1, 2, 3); therefore, they are w-positions, as are (1, 2, 5) and (1, 3, 5). On the other hand (1, 4, 5) is shown by enumeration of possible moves to be an l-position. In general $(1, 2k, 2k + 1)$ is an l-position. Any move and an appropriate countermove change it to $(2k, 2k)$ or (1, 1) or $(1, 2n, 2n + 1)$ where $n < k$.

For triplets in which the smallest number is greater than 1, the enumerative method is hardly practical because of the rapidly increasing number of possible moves and counter moves. A different approach is needed.

Any triplet contains three number-pairs as subsets. Concerning these pairs, we have a sort of "exclusion principle". *If each triplet in a set of triplets is an l-position then any pair can occur only once as a subset of any of the triplets.* Indeed, if (a, b, c) and (a, b, d) were two of the triplets and, say, $d>c$, a single move could change d into c to make the two triplets identical. This shows that (a, b, c) and (a, b, d) cannot both be l-positions.

This remark suggests a procedure for generating increasingly large sets of restricted triplets, with no pair repeated.

First we arrange number pairs in rows with each row containing all pairs that share the same smaller number. Within a row the pairs are in order of their larger numbers. The rows are written one below the other in order of their smaller numbers. The scheme is a rectangular array (see Table 1).

TABLE 1.

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)
(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	
(3, 4)	(3, 5)	(3, 6)	(3, 7)		
(4, 5)	(4, 6)	(4, 7)			
(5, 6)	(5, 7)				
(6, 7)					

In this arrangement pairs to be added to the scheme along a diagonal are all those having the same larger number. We name the successive diagonals according to the larger number in their pairs: e. g., in Table 1, diagonal 7 is just below diagonal 6.

We first illustrate the procedure for generating restricted triplets on the set of pairs above or on diagonal 6. We say two pairs are compatible if their union is a triplet.

We combine the first pair, (1, 2) with the next compatible pair (1, 3). The result is the triplet (1, 2, 3). By the "exclusion principle" pairs (1, 2), (1, 3) and (2, 3) are eliminated from further consideration. The next available pair is (1, 4) and a compatible pair is (1, 5); they form the triplet (1, 4, 5) and pairs (1, 4), (1, 5), (4, 5), are eliminated. For pair (1, 6), however, there is no compatible pair left. It is not eliminated but kept for further consideration when the pair-scheme is enlarged. Proceeding in the same way in the second and subsequent rows we associate pairs (2, 4) and (2, 6), also pairs (3, 5) and (5, 6), obtaining triplets (2, 4, 6) and (3, 5, 6). This ends the triplet-generating process for the pair-set restricted to 6. Four triplets have been formed, and three pairs remain for later consideration.

We continue the process by adding the pairs of diagonal 7. In the same orderly fashion we obtain triplets (1, 6, 7), (2, 5, 7) and (3, 4, 7). In this case *all the pairs* above or on the diagonal have been used and altogether seven triplets have been obtained.

Now we add diagonal after diagonal, always forming all possible triplets before adding the next one. When we reach diagonal 15 (but not before) again *all the pairs* have been used. We call diagonals similar to 7 and 15 *complete* diagonals — a self-explanatory expression.

The next complete diagonal is 31. The set then consists of 155 triplets generated from 465 pairs. For brevity's sake, we call this set of triplets the L_{31} set. A few remarks are in order about this set. Its elements do have some characteristics of *l*-positions. For example a change of any one number of any of the triplets yields a triplet not in the set. More important, the L_{31} set can be used in actual play and force a win if the initial position is an element of L_{31}, and the opponent is to move. Thus the L_{31} set is a set of *l*-positions by experiment. Made confident by this result we turn now to our planned observations on our 155 triplets.

We order the 155 triplets in a way convenient for observing regularities. We write the triplets in order of their smallest number in columns as in Table 2. Within a column the triplets follow each other in order of their middle numbers. The columns themselves are ordered in accordance with the smallest number in their triplets (Table 2).

TABLE 2.

1, 2, 3	2, 4, 6	3, 4, 7	4, 8, 12	5,8, 13	6, 8, 14	7, 8, 15	8, 16, 24
4, 5	5, 7	5, 6	9, 13	9, 12	9, 15	9, 14	17, 25
6, 7	8, 10	8, 11	10, 14	10, 15	10, 12	10, 13	18, 26
8, 9	9, 11	9, 10	11, 15	11, 14	11, 13	11, 12	19, 27
10, 11	12, 14	12, 15	16, 20	16, 21	16, 22	16, 23	20, 28
12, 13	13, 15	13, 14	17, 21	17, 20	17, 23	17, 22	21, 29
14, 15	16, 18	16, 19	18, 22	18, 23	18, 20	18, 21	22, 30
16, 17	17, 19	17, 18	19, 23	19, 22	19, 21	19, 20	23, 31
18, 19	20, 22	20, 23	24, 28	24, 29	24, 30	24, 31	
20, 21	21, 23	21, 22	25, 29	25, 28	25, 31	25, 30	
22, 23	24, 26	24, 27	26, 30	26, 31	26, 28	26, 29	
24, 25	25, 27	25, 26	27, 31	27, 30	27, 29	27, 28	
26, 27	28, 30	28, 31					
28, 29	29, 31	29, 30					
30, 31							

9, 16, 25	10, 16, 26	11,16, 27	12, 16, 28	13, 16, 29	14, 16, 30	15, 16, 31
17, 24	17, 27	17, 26	17, 29	17, 28	17, 31	17, 30
18, 27	18, 24	18, 25	18, 30	18, 31	18, 28	18, 29
19, 26	19, 25	19, 24	19, 31	19, 30	19, 29	19, 28
20, 29	20, 30	20, 31	20, 24	20, 25	20, 26	20, 27
21, 28	21, 31	21, 30	21, 25	21, 24	21, 27	21, 26
22, 31	22, 28	22, 29	22, 26	22, 27	22, 24	22, 25
23, 30	23, 29	23, 28	23, 27	23, 26	23, 25	23, 24

In describing our observations we denote by (a, b, c), $(a < b < c)$ any one of the triplets. Many observations can be made.

We list below only the ones to be used in the following argument.

1. $a + b + c$ is even for any (a, b, c) in L_{31}.
2. If $(2a, 2b, 2c)$ is in L_{31}, so is (a, b, c).
3. If any two numbers in a triplet in L_{31} are odd, another triplet in L_{31} can be obtained by subtracting 1 from the odd numbers.
4. The complete diagonals are those corresponding to the numbers 2^{n-1}, $n = 3, 4, 5$.

We now make the sweeping conjecture that our triplet generating process can be continued indefinitely, producing an increasing sequence of L sets of triplets all of which are l-positions and that our observations are correct for all these sets.

Assuming this conjecture to be true we note that each triplet in an L set must have either one or three even entries. If a triplet has three even entries then dividing the numbers by 2 gives another triplet in the set. If the triplet has only one even entry then it is of the form $(2k+1_1, 2m+1_1, 2n)$ or some permutation of these numbers; subtracting 1 from the odd numbers and dividing the resulting three even numbers by 2 gives another triplet in the set.

We illustrate this on the triplets (25, 43, 50), (16, 39, 47) and (29, 63, 66).

TABLE 3

Example I	*Example II*	*Example III*
25,43,50	16,39,47	29,63,66
12,21,25	8,19,23	14,31,33
6,10,12	4, 9,11	7,15,16
3, 5, 6	2, 4, 5	3, 7, 8
1, 2, 3	1, 2, 2	1, 3, 4
0, 1, 1		0, 1, 2

In Example I, (Table 3) we reach an obvious l-position. In Example II, the sum of the numbers in the last triplet is odd: the starting triplet cannot be an element of an L-set. The same is the case in Example III, the last triplet not being an element of an L-set. In fact, our observations when applied to any triplet, reveal whether the triplet is an element of an L-set or not. In practical playing this is a cumbersome test if the numbers in a triplet are large, and, of course, it is inapplicable to positions with more than three piles.

A fairly obvious trick simplifies the test, making its application more practical, and — most importantly — suggests both a general test of any position and a method of proving its validity. We observe that only two arithmetical operations are involved in the step from one triplet to the next in the descending sequence, namely, division by 2, and subtraction of 1.

If the numbers in a triplet are expressed in the binary system then both operations affect only the last digit of each number. This is so only in the binary system. If the three numbers in a triplet are even the last digit of each is 0 and division by 2 is accomplished by simply deleting the last digit. If just one number in a triplet is even we subtract 1 from each of the other two numbers. Since the last digit is 1 this operation is effected by changing this last digit from 1 to 0 in each of the two numbers.

TABLE 4

Example I			*Example II*		
11001	101011	110010	10000	100111	101111
1100	10101	11001	1000	10011	10111
110	1010	1100	100	1001	1011
11	101	110	10	100	101
1	10	11	1	10	10
0	1	1			

Table 4 shows Examples I and II of Table 3 but with the numbers in binary notation.

The simplification achieved is evident; whether all of the three last digits are 0, or whether only one of them is, the operation yielding the succeeding triplet is to omit the last digits of all three numbers. If eventually a triplet is reached for which the sum of the last digits of the three numbers is odd then the sum of these numbers is odd, and that triplet and all preceding ones are not *l*-positions.

If we write the three binary numbers of the triplet one beneath the other just as if they were to be added, the digits of the same place value form columns. It is then clear that the step-by-step omission of the last digits can be replaced by a one-step survey of the sums of the three digits in the same column to find out whether any of the sums are odd. The test of a triplet then assumes the form: *Write all the numbers of the counters in the piles in the binary system, one beneath the other, as if they were to be added. Obtain all the sums of the digits in each column. The position is a losing one if, and only if, all these sums are even.* (*Table* 5.)

TABLE 5

11001	10000
101011	100111
110010	101111
222022	211222

This is the generally known strategy-rule of Nim. Deductive proof of its correctness is given, e. g., in Hardy and Wright, *loc. cit.* The proof then changes all our conjectures into facts. Furthermore, the rule and its proof are valid for an arbitrary number of piles. It also provides a simple method for changing any winning position into a losing one by a single move.

The author is grateful to the referee for valuable suggestions of improvements and simplifications.

A "GOOD" GENERALIZATION OF THE EULER-FERMAT THEOREM

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The Euler-Fermat Theorem (hereafter denoted EFT) may be stated in the form: *If* $(k,m)=1$, *then* $k^{\phi(m)+1}\equiv k \pmod{m}$. It asserts that power residues of appropriate values of k display periodic behavior and that the period length is not greater than $\phi(m)$.

A table of power residues of the integers 0 through $m-1$, modulo any positive modulus m, illustrates that the power residues of any integer k display periodic behavior without regard to the restriction that $(k,m)=1$. This fact points toward a generalization of EFT.

The ideal generalization of EFT would have the form:

If k is an integer and m is an integer greater than two, then (1) $k^{f(m,k)} \equiv k^{g(m,k)}$ (mod m), where $f(m,k)$ and $g(m,k)$ are the least positive integers such that (1) holds and $g(m,k) < f(m,k)$.
[The author is indebted to the referee for the suggestion of this "best" form of the generalization.]

Weaver [**2**] investigated some of the properties of power residues of all integers modulo a composite modulus, but he gave no generalization of EFT.

T. Szele [**1**, see footnote 2], attributes to L. Rédei the following generalization of EFT:

For all integers k, $k^m \equiv k^{m-\phi(m)}$ (mod m). Rédei's generalization removes the restriction that $(k,m) = 1$. It gives, for all k, that the maximum period length is $\phi(m)$, and that, for each k, periodic behavior is seen no later than at the mth power. As will be seen, however, Rédei's generalization does not achieve the ideal.

Theorem A, given below, provides a generalization of EFT which also removes the restriction that $(k,m) = 1$. From Theorem B we may see that, for each positive composite modulus m, and all integers k, periodic behavior of the power residues is displayed earlier than is shown by Rédei's form—in fact, no later than the power $\phi(m) + 1$—and hence his form is not as "good" as that of Theorem A. For some integers k and m, the form of Theorem A achieves the ideal in that the exponents are minimal; for other values of k, the exponents are not minimal. Hence Theorem A is also not the "best" generalization.

We begin the discussion leading to a proof of Theorem A by listing some notation which will be used throughout.

(1) All letters represent nonnegative integers; let k represent the least positive integer in a given residue class modulo m.

(2) Let $$m = P_1 \cdot Q, \qquad k = P_2 \cdot R,$$

in which

$$P_1 = \prod_{i=1}^{f} p_i^{t_i}, \qquad t_i \geqq 1, f \geqq 1,$$

$$Q = \prod_{i=1}^{g} q_i^{u_i}, \qquad u_i \geqq 0, g \geqq 1,$$

$$P_2 = \prod_{i=1}^{f} p_i^{v_i}, \qquad v_i \geqq 1, f \geqq 1,$$

$$R = \prod_{i=1}^{h} r_i^{w_i}, \qquad w_i \geqq 0, h \geqq 1,$$

and in which, for all admissible values of i, the p_i, q_i, r_i are all distinct primes.

(3) Let b be the smallest positive integer such that $bv_i \geqq t_i$ for all $i = 1, 2, \cdots, f$.

From these definitions and limitations it is readily seen that $(k,Q) = 1$, $(k,P_1) \neq 1$

and hence $(k,m) \neq 1$. Since $(k,Q)=1$, it is known that k belongs to some positive exponent e (modulo Q) and that $e \mid \phi(Q)$.

Weaver proved the following lemma [2, p. 130]:

LEMMA 1. *For b and e as defined above, $k^{b+e} \equiv k^b \pmod{m}$.*

Lemmas 2, 3, and 4 are easily obtained and are stated without proof.

LEMMA 2. *For all $t \geqq 1$ and $p \geqq 2$, $p \leqq p^t(p-1)$, and if $t > 1$, then $p < p^t(p-1)$.*

LEMMA 3. *If $t \geqq 1$ and $p \geqq 2$, then $t \leqq p^{t-1}$.*

LEMMA 4. *If $t \geqq 1$ and $p \geqq 2$, then $t \leqq p^{t-1}(p-1)$.*

LEMMA 5. *For b as defined above, $b \leqq \phi(P_1)$.*

Proof. By definition, b is the least positive integer such that $bv_i \geqq t_i$ for all admissible i. Since $v_i \geqq 1$ for each i, it is obvious that $t_i v_i \geqq t_i$ for each i. Hence b does not exceed the largest t_i. Let the maximum t_i be t_j, $1 \leqq j \leqq f$. Then $b \leqq t_j$. By Lemma 4 and the fact that for each prime p_i, $p_i \geqq 2$, it follows that $t_j \leqq p_j^{t_j-1}(p_j-1)$. Also for each admissible i, $p_i^{t_i-1}(p_i-1) \geqq 1$. Therefore,

$$t_j \leqq \prod_{i=1}^{f} p_i^{t_i-1}(p_i-1) = \phi(P_1).$$

Thus, since $b \leqq t_j$, it follows that $b \leqq \phi(P_1)$.

THEOREM A. *For every integer k and every integer $m \geqq 1$,*

$$k^{\phi(m)+1} \equiv k^{\phi(m)+1-\phi(Q)} \pmod{m},$$

in which Q is the largest factor of m which is relatively prime to k.

Proof. If $m=1$, the result is true trivially, and if $(k,m)=1$, it follows from EFT that

$$k^{\phi(m)+1} \equiv k \equiv k^{\phi(m)+1-\phi(m)} \pmod{m},$$

as claimed.

For the m and k of item (2) of the notation, by Lemma 5, $b \leqq \phi(P_1)$, and since $\phi(Q) \geqq 1$, then

$$b \leqq \phi(Q)\phi(P_1) - \phi(Q) + 1 = \phi(m) + 1 - \phi(Q).$$

Let $b+s = \phi(m)+1-\phi(Q)$ for some $s \geqq 0$. Since, for e as defined above, $e \mid \phi(Q)$, then $\phi(Q) = je$ for some positive integer j. From Lemma 1, $k^b \equiv k^{b+e} \pmod{m}$. Successive multiplications by k^e yield $k^b \equiv k^{b+e} \equiv k^{b+2e} \equiv \cdots \equiv k^{b+je} \pmod{m}$. A final multiplication by k^s yields $k^{b+s} \equiv k^{b+je+s} \pmod{m}$. Substituting for je and $b+s$, the conclusion follows that

$$k^{\phi(m)+1-\phi(Q)} \equiv k^{\phi(m)+1} \pmod{m}.$$

THEOREM B. *For b as defined in* (3), $b \leqq \min\{\phi(m), m - \phi(m)\}$.

Proof. By Lemma 3, $t_j \leqq p_j^{t_j - 1}$. Since the ϕ-function is multiplicative, $\phi(m) = \phi(P_1 Q) = \phi(P_1)\phi(Q)$. Also, since $p_i \geqq 2$, $t_i \geqq 1$ for $i = 1, 2, \cdots, f$, then $p_i^{t_i - 1} \geqq 1$, and

$$\prod_{i=1}^{f} p_i - \prod_{i=1}^{f} (p_i - 1) \geqq 1.$$

Thus, for any j,

$$t_j \leqq \prod_{i=1}^{f} p_i^{t_i - 1} \left[\prod_{i=1}^{f} p_i - \prod_{i=1}^{f} (p_i - 1) \right] \cdot \phi(Q)$$
$$= P_1 \phi(Q) - \phi(P_1)\phi(Q) \leqq P_1 Q - \phi(P_1)\phi(Q) = m - \phi(m).$$

Therefore, $b \leqq t_j \leqq m - \phi(m)$. Also, by Lemma 5, $b \leqq \phi(P_1)$, whence the conclusion of the theorem.

Note that if the prime factorization of k contains all of the prime factors of m (i.e., $u_i = 0$ in the notation of (2)), then $k^b \equiv k^{b+1} \equiv 0 \pmod{m}$, so the period is in fact of length one, and the period begins at b.

In conclusion we make two additional comments. First, in much the same way that in elementary calculus Rolle's Theorem may be used as a lemma in the proof of the mean value theorem, in this paper EFT is used as a foundation in the proof of Theorem A and then may be shown to be a special case of Theorem A. (The proof of this assertion is trivial.)

Second, it should be noted that both Theorem A and Rédei's form follow logically from Lemma 1. The proof that Theorem A follows has been given. Lemma 1 implies that, in the exponentiation of the integer k, a cycle of period e has begun at or before exponent b. By Theorem B, $b \leqq m - \phi(m)$. That is, the residue of $k^{m-\phi(m)}$ is repeated by $k^{m-\phi(m)+ie}$ for all positive integers i, and hence specifically for the integer j of the proof of Theorem A. Hence, $k^{m-\phi(m)} \equiv k^{m-\phi(m)+je} \equiv k^m \pmod{m}$.

References

1. Tibor Szele, Une généralisation de la congruence de Fermat, Matematisk Tidsskrift B, (1948) 57–59.
2. Milo Weaver, Cosets in a semi-group, this MAGAZINE, 25 (1953) 125–136.

A NOTE ON ARITHMETIC PROGRESSIONS OF LENGTH THREE

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Introduction. Let n, k and b be positive integers such that $1 \leqq k, b \leqq n$ $(k \neq b)$. Define $f_n(k, b) =$ the number of integer arithmetic progressions of length three between 1 and n to which k and b belong. Notice that $f_n(k, b) = 0, 1, 2$, or 3. The object of this note is to find asymptotic estimates for the number $t_i(n)$ of pairs (k, b) for which f_n takes the value i ($i = 0, 1, 2, 3$). In the process, we calculate the limit (on n) of the average value of $f_n(k, b)$.

We will use the following notation in this paper [see **1**]. If h and g are two functions defined on the positive integers, we say $h(n) \sim g(n)$ if $\lim_{n\to\infty} (h(n)/g(n)) = 1$ and we read $h(n)$ is asymptotic to $g(n)$.

The results. We begin with the following lemma:

LEMMA. *The number of arithmetic progressions of length three, all of whose members are between* 1 *and* n, *is asymptotic to* $n^2/4$.

Proof. The number we want is $\sum_{k=1}^{n} [(n-k)/2] \sim n^2/4$ since if k is the first term of an arithmetic progression of length three between 1 and n, then precisely the integers $1, 2, \cdots, [(n-k)/2]$ can serve as the difference of the arithmetic progression.

If we count each arithmetic progression between 1 and n six times (once for each pair in the arithmetic progression), then we get, by the lemma, that

$$6 \cdot \frac{n^2}{4} \sim \sum_{\substack{(k,b) \\ 1 \leqq k \neq b \leqq n}} f_n(k, b) \quad \text{i.e.,}$$

$$\sum f_n(k, b) \sim \frac{3n^2}{2}.$$

Now there are $n^2 - n$ pairs (k, b) and hence $a(n)$, average value of $f_n(k, b)$, is asymptotic to $\frac{3}{2}$ (that is, $\lim_{n\to\infty} a(n) = \frac{3}{2}$).

Next, we show that $t_0(n) \sim t_3(n) \sim (n^2/6)$. In each case, the number of pairs counted is clearly twice the number of such pairs which satisfy $k < b$. If $k < b$, then $f_n(k, b) = 0$ if and only if

(1) $n - b < b - k$,

(2) $k < b - k + 1$, and

(3) $b - k$ is odd.

Observe that (1) means there is no integer m such that $b < m \leqq n$ and m is in arithmetic progression with k and b. (2) means that there is no integer m such that $1 \leqq m < k$ and m is in arithmetic progression with k and b. Finally, (3) means that there is no integer m such that $k < m < b$ and m is in arithmetic progression with k and b.

The number of pairs (k, b) for which $f_n(k, b) = 0$ and $k < b$ can be found as follows: If $k \geqq (n+2)/3$, then precisely those b's satisfying $b > 2k - 1$ and $b - k$ odd give us a pair (k, b) for which $f_n(k, b) = 0$ and $k < b$, because for $k \geqq (n+2)/3$ we have (2) implies (1). If $k < (n+2)/3$, then precisely those b's satisfying $b > (n+k)/2$ and $b - k$ odd give us a pair (k, b) for which $f_n(k, b) = 0$ and $k < b$, because for $k < (n+2)/3$ we have (1) implies (2). Now, remembering also to count the pairs (k, b) with $k > b$, we see that the number of pairs (k, b) for which $f_n(k, b) = 0$ is asymptotic to

$$2\left(\frac{1}{2}\sum_{k=1}^{(n+2)/3}\left(n-\frac{n+k}{2}\right)+\frac{1}{2}\sum_{k=(n+2)/3}^{(n+1)/2}(n-(2k-1))\right)\sim\frac{n^2}{6}.$$

Also, if $k < b$, then $f_n(k,b) = 3$ if and only if

(4) $n - b \geqq b - k > 0$,

(5) $b > k \geqq b - k + 1$, and

(6) $b - k$ is even.

Observe that (4) means that there is an integer m such that $b < m \leqq n$ and m is in arithmetic progression with k and b. (5) means that there is an integer m such that $1 \leqq m < k$ and m is in arithmetic progression with k and b. Finally, (6) means that there is an integer m such that $k < m < b$ and m is in arithmetic progression with k and b.

The number of pairs (k, b) for which $f_n(k,b) = 3$ and $k < b$ can be found as follows: If $k \leqq (n+2)/3$, then precisely those b's for which $k < b \leqq 2k - 1$ and $b - k$ is even give us a pair (k,b) for which $f_n(k,b) = 3$ and $k < b$, because for $k \leqq (n+2)/3$ we have (5) implies (4). If $k > (n+2)/3$, then precisely those b's for which $k < b \leqq (n+k)/2$ and $b - k$ is even give us a pair (k,b) for which $f_n(k,b) = 3$ and $k < b$, because for $k > (n+2)/3$ we have (4) implies (5). Now, remembering also to count the pairs (k,b) with $k > b$, we see that the number of pairs (k,b) for which $f_n(k,b) = 3$ is asymptotic to

$$2\left(\frac{1}{2}\sum_{k=1}^{(n+2)/3}(2k-1-k)+\frac{1}{2}\sum_{k=(n+2)/3}^{n}\left(\frac{n+k}{2}-k\right)\right)\sim\frac{n^2}{6}.$$

Finally, we find asymptotic estimates for $t_1(n)$ and $t_2(n)$. We have

$$a(n) = \frac{0\cdot t_0(n)+1\cdot t_1(n)+2\cdot t_2(n)+3\cdot t_3(n)}{n^2-n}\sim\frac{t_1(n)+2t_2(n)+n^2/2}{n^2-n}.$$

Since $\lim_{n\to\infty}a(n) = \frac{3}{2}$, it follows that $t_1(n) + 2\,t_2(n) \sim n^2$. On the other hand we also have

$$n^2 - n = t_0(n) + t_1(n) + t_2(n) + t_3(n),$$

and since

$$t_0(n) \sim t_3(n) \sim \frac{n^2}{6},$$

it follows that

$$t_1(n) + t_2(n) \sim \frac{2n^2}{3}.$$

Hence we have

$$t_1(n) \sim t_2(n) \sim \frac{n^2}{3}.$$

Summarizing our results we have the following theorem:

THEOREM. *Let $t_i(n)$ be the number of pairs (k,b), $1 \leqq k,\ b \leqq n,\ k \neq b$ for which $f_n(k,b) = i$ $(i = 0,1,2,3)$ and let $a(n)$ be the average value of $f_n(k,b)$ over all pairs (k,b), $1 \leqq k,b \leqq n$, $k \neq b$. Then* $\lim_{n\to\infty} a(n) = 3/2$, $t_0(n) \sim t_3(n) \sim n^2/6$ *and* $t_1(n) \sim t_2(n) \sim n^2/3$.

It would be interesting to see a treatment of this problem for arithmetic progressions of length greater than 3.

Reference

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 1960.

A THEOREM ON AN INSCRIBED TRIANGLE

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We consider a triangle $A_1A_2A_3$ with B_1 between A_2 and A_3, B_2 between A_3 and A_1, B_3 between A_1 and A_2. Let the areas of $A_1A_2A_3$, $B_2A_1B_3$, $B_3A_2B_1$, $B_1A_3B_2$ and $B_1B_2B_3$ be F, F_1, F_2, F_3 and G respectively (Figure 1). Rigby [1] has proved the inequality

$$G^3 + (F_1 + F_2 + F_3)G^2 - 4F_1F_2F_3 \geqq 0. \tag{1}$$

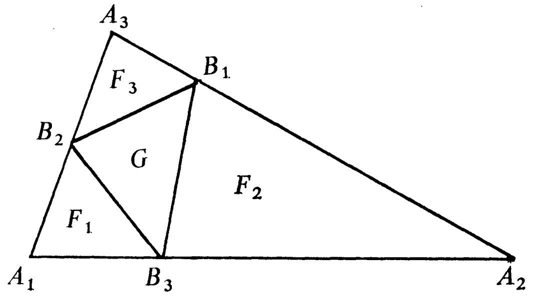

FIG. 1

We derive an inequality for F, F_1, F_2, F_3, thus answering the question: *What is the minimum area F of a triangle $A_1A_2A_3$ in which it is possible to cut off, in the described manner, three triangles with given areas F_1, F_2, F_3?*

A first condition is obviously

$$F > F_1 + F_2 + F_3. \tag{2}$$

Furthermore, putting

$$F_1 + F_2 + F_3 = 3M, \ F_1F_2F_3 = P \tag{3}$$

and noting that $G = F - 3M$, we obtain from (1)

$$f(F) \equiv F^3 - 6MF^2 + 9M^2F - 4P \geqq 0. \tag{4}$$

In order to solve the equation $f(F) = 0$, we write $F = x + 2M$, obtaining

$$g(x) \equiv x^3 + px + q = 0, \tag{5}$$

with

$$p = -3M^2, \ q = 2(M^3 - 2P). \tag{6}$$

For the discriminant of (5) we have

$$D = \frac{1}{4}q^2 + \frac{1}{27}p^3 = 4P(P - M^3) \leqq 0, \tag{7}$$

in view of the theorem on the arithmetical and the geometrical mean. Hence $g(x) = 0$ has three real roots. Solving by the Cardanus method we obtain

$$x_i = 2M\cos\phi_i, \qquad (i = 1,2,3) \tag{8}$$

with

$$\cos 3\phi_i = (2P - M^3)/M^3, \tag{9}$$

and therefore for the roots R_i of $f(F) = 0$,

$$R_i = 2M(1 + \cos\phi_i). \tag{10}$$

All three are positive. Let $R_1 \leqq R_2 \leqq R_3$. A graph of $f(F)$ is given in Figure 2. We have $f(0) = f(3M) = -4P$, $f(M) = 4(M^3 - P) \geqq 0$, $f'(M) = f'(3M) = 0$. Hence $R_1 \leqq R_2 < 3M < R_3$. In view of (2) and (4) it is necessary that $F \geqq R_3$ and Rigby's argument (his Theorem 2) shows this condition to be sufficient.

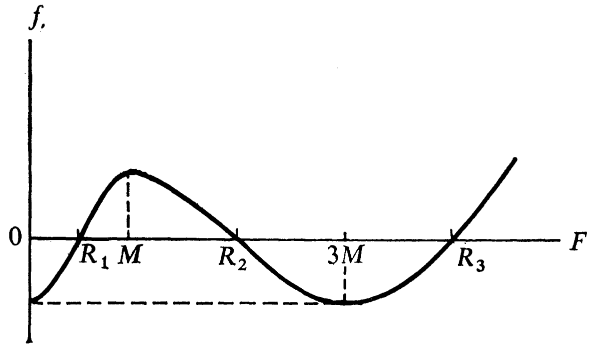

FIG. 2

Equation (9) with $3\phi_i = \alpha$ gives one value α_1 with $0 < \alpha_1 < \pi$. If we take $\phi_3 = \alpha_1/3$, $\phi_2 = (\alpha_1 + 4\pi)/3$, $\phi_1 = (\alpha_1 + 2\pi)/3$ it is easy to verify that in all cases ($2P - M^3 > 0, = 0, < 0$) we have $R_1 \leqq R_2 < R_3$.

Summarizing we have the theorem: *In a triangle $A_1A_2A_3$ with area F we can inscribe a triangle $B_1B_2B_3$ such that the remaining triangles have the given areas F_1, F_2, F_3 if and only if*

$$F \geqq F_M = 2M \left[1 + \cos\left\{\frac{1}{3}\arccos\left(\frac{2P}{M^3} - 1\right)\right\}\right], \tag{11}$$

M and P being given by (3). It follows from Rigby's paper that if the condition is satisfied there are two inscribed triangles with the property.

For a fixed value of M the minimum F_M is an increasing function of P; as $0 < P \leqq M^3$, we have

$$3M < F_M \leqq 4M. \tag{12}$$

We add some examples. For $F_1 = F_2 = F_3 = M$, we have $R_1 = R_2 = M$, $R_3 = F_M = 4M$; $2P = M^3$ implies $R_1 = (2 - \sqrt{3})\,M$, $R_2 = 2M$, $R_3 = F_M = (2 + \sqrt{3})M = 3.73M$; for $F_1\colon F_2\colon F_3 = 3\colon 4\colon 5$ we have $F_M = 3.97\,M$; for $F_1\colon F_2\colon F_3 = 1\colon 5\colon 6$ we find $F_M = 3.71M$; and for $F_1\colon F_2\colon F_3 = 1\colon 9\colon 20$ the minimum F_M is $3.46M$.

Reference

1. J. F. Rigby, Inequalities concerning the areas obtained when one triangle is inscribed in another, this MAGAZINE, 45 (1972) 113–116.

A RELATIONSHIP BETWEEN AN INTEGER AND THE ONE WITH THE REVERSED ORDER OF DIGITS

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The 1971 William Lowell Putnam Mathematical Competition contained the following problem (question B-6):

Let $\delta(x)$ be the greatest odd divisor of the positive integer x. Show that

$$\left| \sum_{n=1}^{x} \frac{\delta(n)}{n} - \frac{2}{3}x \right| < 1, \tag{1}$$

for all positive integers x.

One of the results of this paper will be to show that (1) can be improved to

$$0 < \sum_{n=1}^{x} \frac{\delta(n)}{n} - \frac{2}{3}x < \frac{2}{3}. \tag{2}$$

This can also be written as

$$0 < \sum_{n=1}^{x} \frac{1}{r_2(n)} - \frac{2}{3}x < \frac{2}{3}, \tag{3}$$

where we define $r_2(n) = 2^k$, where k is the largest exponent for which 2^k divides n.

We will show that (3) can be generalized to any positive integer $a \geqq 2$ as follows:

$$0 < \sum_{n=1}^{x} \frac{1}{r_a(n)} - \frac{a}{a+1}x < \frac{a}{a+1}, \tag{4}$$

where we define $r_a(n) = a^k$, where k is the largest exponent for which a^k divides n.

Our main result is a theorem giving an exact value of the difference appearing in (4) in terms of the integer whose digits in base a are in the reverse order of that of x. From this theorem, (4) follows as a corollary.

THEOREM. *Let the representation of a positive integer x in base a be $x = (b_r \cdots b_0)_a$. Then*

$$\sum_{n=1}^{x} \frac{1}{r_a(n)} - \frac{ax}{a+1} = \frac{(b_0 \cdots b_r)_a}{(a+1)a^r}. \tag{5}$$

In order to prove this theorem, we need the following

LEMMA. *For $a \geqq 2$, $x \geqq 1$, let*

$$f_a(x) = \sum_{n=1}^{x} \frac{1}{r_a(n)} - \frac{ax}{a+1},$$

then

$$f_a(ax) = \frac{1}{a} f_a(x).$$

Proof of the lemma.

$$f_a(x) + \frac{ax}{a+1} = \sum_{n=1}^{x} \frac{1}{r_a(n)} = \sum_{r=0}^{\infty} \sum_{\substack{1 \leqq n \leqq x \\ r_a(n) = a^r}} \frac{1}{a^r},$$

where the inner summation is over all integers n for which $1 \leqq n \leqq x$ and $r_a(n) = a^r$. Similar notation will be used in the next line. Therefore,

$$f_a(x) + \frac{ax}{a+1} = \sum_{r=0}^{\infty} \left(\sum_{\substack{1 \leqq n \leqq x \\ a^r | n}} \frac{1}{a^r} - \sum_{\substack{1 \leqq n \leqq x \\ a^{r+1} | n}} \frac{1}{a^r} \right) = \sum_{r=0}^{\infty} \frac{\left[\frac{x}{a^r}\right] - \left[\frac{x}{a^{r+1}}\right]}{a^r}.$$

Note that

$$\sum_{r=0}^{\infty} \frac{\frac{x}{a^r} - \frac{x}{a^{r+1}}}{a^r} = \sum_{r=0}^{\infty} x \cdot \frac{a-1}{a^{2r+1}}$$

$$= \frac{(a-1)x}{a} \cdot \sum_{r=0}^{\infty} \frac{1}{a^{2r}} = \frac{(a-1)x}{a} \cdot \frac{a^2}{a^2-1} = \frac{ax}{a+1}.$$

Thus

$$f_a(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{a^{r+1}} - \left[\frac{x}{a^{r+1}}\right]\right) - \left(\frac{x}{a^r} - \left[\frac{x}{a^r}\right]\right)}{a^r}$$

and

$$f_a(ax) = \sum_{r=0}^{\infty} \frac{\left(\frac{ax}{a^{r+1}} - \left[\frac{ax}{a^{r+1}}\right]\right) - \left(\frac{ax}{a^r} - \left[\frac{ax}{a^r}\right]\right)}{a^r}$$

$$= \sum_{r=0}^{\infty} \frac{\left(\frac{x}{a^{r}} - \left[\frac{x}{a^{r}}\right]\right) - \left(\frac{x}{a^{r-1}} - \left[\frac{x}{a^{r-1}}\right]\right)}{a^r}$$

$$= \sum_{r=-1}^{\infty} \frac{\left(\frac{x}{a^{r+1}} - \left[\frac{x}{a^{r+1}}\right]\right) - \left(\frac{x}{a^r} - \left[\frac{x}{a^r}\right]\right)}{a^{r+1}}$$

$$= \frac{\left(\frac{x}{a^{-1+1}} - \left[\frac{x}{a^{-1+1}}\right]\right) - \left(\frac{x}{a^{-1}} - \left[\frac{x}{a^{-1}}\right]\right)}{a^{-1+1}}$$

$$+ \frac{1}{a} \sum_{r=0}^{\infty} \frac{\left(\frac{x}{a^{r+1}} - \left[\frac{x}{a^{r+1}}\right]\right) - \left(\frac{x}{a^r} - \left[\frac{x}{a^r}\right]\right)}{a^r}$$

$$= (x - [x]) - (ax - [ax]) + \frac{1}{a} f_a(x) = \frac{1}{a} f_a(x),$$

as required.

Proof of the theorem. We use induction on x.

If $x = 1$, then

$$f_a(x) = \frac{1}{r_a(1)} - \frac{a}{a+1} = 1 - \frac{a}{a+1} = \frac{1}{a+1}.$$

The representation in base a of 1 is $1 = (1)_a$, so $r = 0$ and $b_0 = 1$. Thus

$$\frac{(b_0 \cdots b_r)_a}{(a+1)a^r} = \frac{(1)_a}{(a+1)a^0} = \frac{1}{a+1} = f_a(1),$$

as required.

Assume the statement of the theorem is true for all x with $1 \leqq x < y$, where y is some integer $\geqq 2$. We consider 2 cases.

Case 1: $a \mid y$. Let $y/a = (c_s \cdots c_0)_a$. Since $1 \leqq y/a < y$,

$$f_a\left(\frac{y}{a}\right) = \frac{(c_0 \cdots c_s)_a}{(a+1)a^s}.$$

Also $y = (b_r \cdots b_0)_a$, where $r = s+1$,

$$b_r = c_s,\ b_{r-1} = c_{s-1}, \cdots, b_1 = c_0,\ b_0 = 0.$$

Then

$$\frac{(b_0 \cdots b_r)_a}{(a+1)a^r} = \frac{(0c_0 \cdots c_s)_a}{(a+1)a^{s+1}} = \frac{1}{a} \cdot \frac{(c_0 \cdots c_s)_a}{(a+1)a^s} = \frac{1}{a} f_a\left(\frac{y}{a}\right).$$

By the lemma, $\frac{1}{a} f_a\left(\frac{y}{a}\right) = f_a(y)$, so the theorem also holds for $x = y$.

Case 2: $a \nmid y$. Let $y - 1 = (c_r \cdots c_0)_a$. Then

$$f_a(y-1) = \frac{(c_0 \cdots c_r)_a}{(a+1)a^r}.$$

Also, $y = (b_r \cdots b_0)_a$, where

$$b_r = c_r, \cdots, \quad b_1 = c_1, \quad b_0 = c_0 + 1.$$

Then

$$\frac{(b_0 \cdots b_r)_a}{(a+1)a^r} = \frac{((c_0+1)c_1 \cdots c_r)_a}{(a+1)a^r} = \frac{a^r + (c_0 \cdots c_r)_a}{(a+1)a^r} = \frac{1}{a+1} + \frac{(c_0 \cdots c_r)_a}{(a+1)a^r}$$
$$= \frac{1}{a+1} + f_a(y-1).$$

But

$$f_a(y) = \sum_{n=1}^{y} \frac{1}{r_a(n)} - \frac{ay}{a+1} = \frac{1}{r_a(y)} - \frac{a}{a+1} + \left(\sum_{n=1}^{y-1} \frac{1}{r_a(n)} - \frac{a(y-1)}{a+1}\right)$$
$$= 1 - \frac{a}{a+1} + f_a(y-1) = \frac{1}{a+1} + f_a(y-1),$$

so the theorem holds for $x = y$, and the proof is complete.

COROLLARY. *If* $x \geqq 1$, *then* $0 < \frac{1}{r_a(n)} - \frac{a}{a+1} x < \frac{a}{a+1}$.

Proof. The first inequality follows immediately from the theorem. Also, if $x = (b_r \cdots b_0)_a$, then

$$f_a(x) = \frac{(b_0 \cdots b_r)_a}{(a+1)a^r} < \frac{a^{r+1}}{(a+1)a^r} = \frac{a}{a+1},$$

as required.

ON A FIXED POINT THEOREM FOR COMPACT METRIC SPACES

D. G. BENNETT and B. FISHER, University of Leicester

A mapping T of a complete metric space (X,ρ) into itself is said to be a contraction mapping if and only if there is a real number c, with $0 < c < 1$, such that

$$\rho(Tx, Ty) \leqq c\rho(x, y) \tag{1}$$

for all x,y in X.

For such a mapping T the following theorem holds:

THEOREM 1. *If T is a contraction mapping on a complete metric space (X, ρ), then there exists a unique point z in X such that $Tz = z$, the point z being called a fixed point.*

If now T is a mapping of a complete metric space (X, ρ) into itself satisfying the weaker condition

$$\rho(Tx, Ty) < \rho(x, y), \qquad (x \neq y) \tag{2}$$

then there does not necessarily exist a fixed point in X. For if (X, ρ) is the set of real numbers with $\rho(x, y) = |x - y|$ and T is defined by

$$Tx = \pi/2 + x - \arctan x,$$

then it is easily seen that T has no fixed point and

$$\begin{aligned} \rho(Tx, Ty) &= |Tx - Ty| \\ &= |1 - (1+\zeta^2)^{-1}|\,|x - y|, \qquad (x \leqq \zeta \leqq y) \\ &= \frac{\zeta^2}{1+\zeta^2}|x-y| \quad < \quad \rho(x, y), \end{aligned}$$

on using the mean value theorem.

However, in a paper by Edelstein [1] he proves the following interesting theorem:

THEOREM 2. *Let T be a mapping of the compact metric space (X,ρ) into itself, satisfying inequality* (2). *Then there exists a unique point z in X such that $Tz = z$.*

Edelstein's proof of this theorem is rather formidable and Kantorovich and Akilov [2] give a fallacious proof by reasoning that such a mapping must be a contraction mapping. It is easily seen that T is not necessarily a contractionm apping by considering the metric space (X,ρ), where $X = \{0,1,\frac{1}{2},\cdots,1/n,\cdots\}$ and $\rho(x,y) = |x-y|$ for x,y in X. Define a mapping T on X by putting

$$T(0) = 0, \quad T(1/n) = 1/(n+1) : n = 1,2,\cdots.$$

Then it is easily seen that T satisfies inequality (2) but not inequality (1) and so T is not a contraction mapping.

We now give a simple proof of this theorem.

Proof of Theorem 2. Let x be an arbitrary point in X. Then since X is compact there exists a point z in X with the property that for arbitrary $\varepsilon > 0$ there exist integers $m, r > 0$ such that

$$\rho(T^m x, z),\ \rho(T^{m+r} x, z) < \varepsilon.$$

Hence

$$\begin{aligned} \rho(z, Tz) &\leqq \rho(z, T^{m+r}x) + \rho(T^{m+r}x, T^r z) + \rho(T^r z, T^{r+1}z) \\ &\quad + \rho(T^{r+1}z, T^{m+r+1}x) + \rho(T^{m+r+1}x, Tz) \\ &< \varepsilon + \rho(T^m x, z) + \rho(Tz, T^2 z) + \rho(z, T^m x) + \rho(T^{m+r}x, z) \\ &< \rho(Tz, T^2 z) + 4\varepsilon. \end{aligned}$$

Since ε is arbitrary it follows that $\rho(z, Tz) \leqq \rho(Tz, T^2 z)$. But since $\rho(Tz, T^2 z) < \rho(z, Tz)$, unless $Tz = z$, we must indeed have $Tz = z$, proving the existence of a fixed point z. The proof that z is unique is trivial.

References

1. M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc., 37 (1962) 74–79.
2. L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces, Pergamon Press, New York, 1964.

VARIATIONS ON CONTINUITY: SETS OF INFINITE LIMIT

R. BUMCROT, Hofstra University and M. SHEINGORN, National Bureau of Standards

1. Let R denote the real numbers and let f be a function from R to R. A formal version of the statement that f is continuous on R is:

(1) $$\forall x \forall \varepsilon \exists \delta \forall t\ 0 \neq |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon.$$

Here it is to be understood that Greek letters vary only over the positive reals. It is amusing and instructive to consider various permutations and modifications of (1). For instance,

(2) $$\forall \varepsilon \exists \delta \forall x \forall t\ \ 0 \neq |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$$

is the statement that f is uniformly continuous on R. It would be an ambitious project indeed to examine all 1536 statements obtainable by (a) permuting the first occurrences of x, t, ε, δ; (b) switching existential and universal quantifiers; or (c) replacing $<$ by $>$. Of course the labor could be greatly reduced by a few simple observations. In the next section we give three examples of "scrambled" continuity statements, and in the last section we prove a theorem that was suggested by another such statement.

2. The statement

$$\forall\varepsilon\forall\delta\forall x\forall t\, 0 \neq |x-t| < \delta \Rightarrow |f(x)-f(t)| < \varepsilon \tag{3}$$

is easily seen to be equivalent to the statement that f is a constant function.

The statement

$$\exists\varepsilon\forall\delta\forall x\forall t\, 0 \neq |x-t| < \delta \Rightarrow |f(x)-f(t)| < \varepsilon \tag{4}$$

characterizes bounded functions. For (4) implies $|f(t)| < |f(0)| + \varepsilon$ for all t, and if $|f(t)| < M$ for all t then (4) holds for $\varepsilon = 2M$.

The statement

$$\exists\varepsilon\forall\delta\forall x\forall t\, 0 \neq |x-t| < \delta \Rightarrow |f(x)-f(t)| > \varepsilon \tag{5}$$

does not hold for any function under consideration (recall f is any function from R to R). For let P be a partition of R into intervals of length $\varepsilon/2$. Since R is uncountable and P is countable, there exist $x \neq t$ in R with $f(x)$ and $f(t)$ in the same member of P, contradicting (5).

3. The statement

$$\forall x\forall\varepsilon\exists\delta\forall t\, 0 \neq |x-t| < \delta \Rightarrow |f(x)-f(t)| > \varepsilon \tag{6}$$

is equivalent to the statement

$$\lim_{t\to x} |f(t)| = +\infty \text{ for all } x,$$

which is easily seen to hold for no function of the type being considered. This suggests the following question: A *set of infinite limit* is a subset A of R such that there exists a function f from R to R for which $\lim_{x\to a} f(x) = +\infty$ for all a in A. What are these sets?

THEOREM. *A is a set of infinite limit if and only if A is a countable G_δ set, i.e., the intersection of a countable set that is a collection of open sets.*

Proof. Suppose A is a set of infinite limit. If A is uncountable then for some M the set $B = \{x \in A : f(x) \leqq M\}$ is uncountable. By a well-known theorem for second-axiom spaces, B contains a limit point b of B ([**1**], p. 48). Then $\lim_{x\to b} f(x) \neq +\infty$, contradicting $b \in A$. Thus A is countable, say $A = \{a_1, a_2, \cdots\}$. For each i and each positive integer n, there exists an open neighborhood O_{in} of a_i such that $f(x) > n$ for all $x \neq a_i$ in O_{in}. For each n the set $U_n = \bigcup_{i=1}^{\infty} O_{in}$ is open, and $A \subset \bigcap_{n=1}^{\infty} U_n$. Suppose $t \in \bigcap_{n=1}^{\infty} U_n$, $t \notin A$. Choose a positive integer $N > f(t)$. Now $t \in U_N$ so $t \in O_{iN}$ for some i. Also $t \neq a_i$, since $t \in A$. Hence $f(t) > N$, a contradiction. Thus, $A = \bigcap_{n=1}^{\infty} U_n$ is a G_δ set.

To show the converse, let A be a countable G_δ set; say $A = \{a_1, a_2, \cdots\} = \bigcap_{i=1}^{\infty} O_i$, where for each i O_i is open. Let $U_0 = R$, $U_n = \bigcap_{i=1}^{n} O_i$, $n \geqq 1$. Define f by: $f(x) = n$ if $x \in U_{n-1} \sim U_n$ for some $n \geqq 1$, $f(a_k) = k$. Clearly f is defined on R. Let $a \in A$; say $a = a_j$. Let K be a positive integer. Choose an open set U containing a_j that does not contain a_i for $1 \leqq i \leqq K$, $i \neq j$. Let $V = U \cap U_K$. Then V is an open neighborhood of a_j and $f(x) \geqq K$ for all $x \neq a_j$ in V. Thus $\lim_{x \to a} f(x) = +\infty$, and A is a set of infinite limit.

Remarks.

(1) A countable G_δ set is nowhere dense. The theorem then implies that a set of infinite limit is nowhere dense, which is exactly what intuition leads one to expect.

(2) A slight modification of this technique would enable one to construct an f which has infinite limit *exactly* on A. Our construction leaves open the possibility that f has infinite limit on a larger (countable) G_δ set.

The second author wishes to thank Dr. Seymour Haber for some useful discussions.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, New Jersey, 1955.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use.*

Applications of Calculus in Business and Economics. By Howard E. Thompson. Benjamin, Reading, Mass., 1972. xiii + 492 pp.

This book was written to be used by students of business and economics in conjunction with a course in calculus. Specifically, the chapters parallel the organization of the *Green Book of Calculus* by Joshua Chover but, with certain reservations to be noted below, it could be used in conjunction with any calculus text.

The book appears to be a real pioneering effort in its field and fills a gap whose existence in the 1960's and 1970's could only be described as disgraceful. Most calculus texts currently available confine themselves to very traditional applications of the calculus and it is past time for a change in this tendency.

In a carefully written preface and an introductory chapter, Professor Thompson explains exactly why mathematics, including calculus, is important in business and economics. Every instructor who will teach from the book and every student who studies it should read this material carefully. In the remaining 17 chapters the author discusses a wide variety of specific applications. Herein the student will find explanations of a variety of basic concepts. Chapter 2 presents the notions of demand, cost and profit functions. In example 2, pages 25–30, for example, he describes a very realistic situation. Chapter 3 takes up the subject of quality inspection. Chapter 4 is devoted to geometric ideas in economics and introduces the concept of the multiplier in national income. In chapter 5 the author returns to the role of functions in economics with a deeper discussion of demand and supply functions. Stocks and bonds are explained in chapter 6. The concept of margin is the subject matter of chapter 7. Here the theory of the firm is first mentioned. Chapter 8 is devoted to the fundamental ideas in equilibrium theory. The remaining chapters 9 through 18 are devoted to more sophisticated applications relying upon the fundamental concepts introduced in the first eight chapters. The author touches upon problems in inventory, capital budgeting, macroeconomics, the theory of the firm, economic growth, the stock market, and decision theory.

In general, the applications are developed with care and the examples are very realistic and representative of problems of the "real" world of economics and business. Moreover, the author has included some good exercises for the student.

The discussion to this point summarizes the reviewer's favorable impressions of the text. I have a few minor criticisms and suggestions for improvement. But first I have a neutral remark which applies to this text as well as several others in my own experience.

Most instructors in calculus who wish to use this material will have to work a little bit to master it themselves. This is because the applications are realistic and therefore relatively complicated. Concepts from the worlds of business and economics must be mastered. It is my own experience and the experience of others that the overwhelming majority of mathematicians is simply not willing to do the extra work this text will require. This is a sad but true fact of life at the present time. Therefore, I predict, in the sincere hope that I will be proven wrong, that this excellent and pioneering text will be a complete failure commercially.

Finally, I have a few relatively minor criticisms and suggestions. First: the author has nowhere drawn a careful distinction between the concepts of comparative statics and dynamics in economics. This may leave some students (and instructors) confused. Second: the author has nowhere given further readings and references. I believe the book would be enhanced by repairing this omission. Third: it would help to make the book more independent of the text by Chover if each chapter began with a brief summary of the concepts from calculus needed therein. This need not lengthen the text excessively and the improvement would be well worth the trouble.

JOHN S. MAYBEE, University of Colorado, Boulder

Fundamental Mathematics, *Unit* 3. *College Algebra*. By James A. Streeter and Gerald N. Alexander. Harper and Row, New York, 1972. $137.00.

This audio-visual program is described by its authors as a "program in College Algebra, organized into 14 modules... designed to cover all of the standard topics taught in a course at this level. Each module consists of a set of film-tape materials with a coordinated section in the study guide. New material is introduced through the film-tape portion of the program with problems at frequent intervals. The study guide provides a concise review of the concepts, further exercises, and practice and mastery tests for each module." The 14 modules include: Sets and Related Topics; Number Systems; Algebraic Operations; Rational Algebraic Expressions; Relations and Functions; Linear Equations and Functions; Quadratic Equations and Functions; Inequalities; Systems of Equations; Theory of Equations; Exponents and Radicals; Matrices and Determinants; Sequences, Series, and the Binomial Theorem; and Exponential and Logarithmic Functions. Since most of these topics are included in a modern eleventh-year algebra course, many college freshmen may find the program most valuable as a review.

The greatest strength of the program is its clarity. For the most part, the material is well organized and meticulously presented. Particularly valuable are the series of worked-out problems which we find in many of the tapes. Structurally, the program is quite solid, although this reviewer would have preferred to see conic sections treated in a separate module rather than as a subordinate topic in the module on systems of equations. Pedagogically, however, the treatment is somewhat threadbare. The material is *told* to the student, rather than *taught* to him; we look in vain for comparisons, analogies, and other devices to promote understanding and retention. Moreover, many important topics are covered so briefly that they would be useful only to students already familiar with them. Perhaps what is needed is a substantial text. Such a text could provide a richer treatment of the material, including proofs (of which virtually none appear here; where proofs are indicated, the user is referred to other sources), and far more practice material. At present, problems are limited to a very few exercises scattered through the study guide.

Of any audio-visual resource, we have the right to ask, "How does this material justify the extra trouble and expense of using it?" The sound track of this program, narrated by a man and woman in turn, certainly makes a real contribution. Despite a few lapses in pronunciation and phrasing, the readings are cordial and intelligent; material heard in this way is undoubtedly more comprehensible than the same passages on the printed page. However, the text itself is a little stiff, as if the authors had trouble striking the proper balance between the formality of the written word and the colloquialism of normal speech. Moreover, the visuals are a disappointment. They consist of little more than mathematical or verbal statements, along with diagrams. Of course, in a program of this type we do not advocate ornament for its own sake. Nonetheless, it is possible to use layout, graphics, illustrations and color to underscore important mathematical points. In these filmstrips, such opportunities are bypassed.

The authors themselves point out that "the program will be best used in conjunction with regular classroom sessions or conferences with mathematical instructors." Employed in this way, it may provide a convenient reference for students wishing to brush up on topics learned elsewhere.

MIRIAM HECHT, Hunter College of the CUNY

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before June 1, 1974.

PROPOSALS

887. *Proposed by Norman Schaumberger, Bronx Community College.*

One solution of the equation $x^y y^x = 1$ is $x = 4$, $y = -2$. Show that all unequal rational solutions are opposite in sign.

888. *Proposed by E. M. Clarke, Madison College, Harrisonberg, Virginia.*

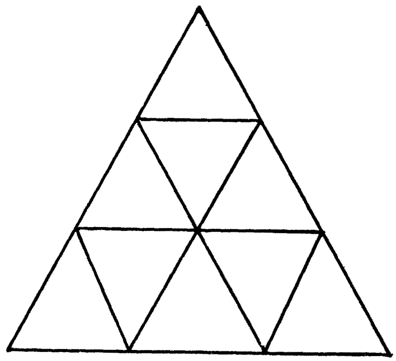
Let G be defined by

$$G = \begin{cases} x - 10 & \text{if } x > 100 \\ G(G(x+11)) & \text{if } x \leqq 100. \end{cases}$$

Compute $\int_0^{100} G(x)dx$.

889. *Proposed by Ralph E. Edwards, Baltimore Life Insurance Company, Maryland.*

In the accompanying diagram, $f(x)$, the different number of triangles, is 13, where x, the number of horizontal lines, is 3. Find the formula for $f(x)$.



890. *Proposed by Zalmin Usiskin, University of Chicago.*

Let a and b be any complex numbers, $a \neq b$. For what a and b does there exist an operator * in c^2 such that $c - \{a\}$ is a group under * with b as the identity?

891. *Proposed by John M. Howell, Littlerock, California.*

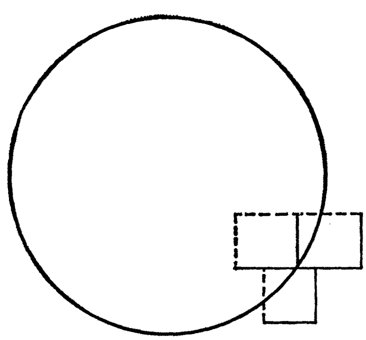
Teams A and B play a series of baseball games. Team A never swings the bat nor steals a base. Team B pitchers never hit a batter and throw strikes half the time. What is the average number of runs scored per game by Team A?

892. *Proposed by Arnold Good, Lewis-St. Frances College, Illinois.*

Let S be a closed set in R^2, a circle will do; and let T be a disjoint collection of half-open intervals in R^2 covering S and satisfying:

1. Every interval of T contains at least one point of S.

2. Every point of S is interior to the union of, at most, four intervals of T (see figure).



Show that T contains only a finite number of intervals.

893. *Proposed by John Herschel, Mission Beach, California.*

Prove that the sum of twin primes whose sum is greater than or equal to 12 is always divisible by 12.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q583. If f is differentiable on $[0,1]$, $f(0) = 0$ and $a > 0$, prove that there exists c in $(0,1)$ such that $f'(c) = ac^{a-1}f(c)/(1-c^a)$.

[*Submitted by Erwin Just*]

Q584. If A, B, C are nonnegative angles satisfying the triangle inequality and with a sum $\leqq \pi$ show that

$$2\sum \sin^2 B \sin^2 C \geqq 4\prod \sin^2 A + \sum \sin^4 A$$

with equality iff $A + B + C = \pi$.

[*Submitted by Murray S. Klamkin*]

Q585. Lucky Larry, a mathematics student whose plausible mistakes in computation always result in correct answers, once wrote an answer in the form

$$a^b c^a = abca$$

where $abca$ represents a four-digit integer whose digits a, b and c are all different. What specific number did Lucky Larry write?

[*Submitted by Alan Wayne*]

Q586. Prove that

$$\sum_{t=0}^{n} \frac{(-1)^t}{m+t+1}\binom{n}{t} = \sum_{t=0}^{m} \frac{(-1)^t}{n+t+1}\binom{m}{t}.$$

[*Submitted by Ralph Garfield*]

Q587. If p is an odd prime different from 5 then either $p^2 - 1$ or $p^2 + 1$ is divisible by 10.

[*Submitted by J. D. Baum*]

(Answers on page 60)

Late Solutions

M. G. Greening, University of New South Wales, Australia: **854, 855**; *Ranee Gupta, University-Liggett School, Michigan:* **854, 855**; *Richard Kerns, Hamburg, Germany:* **854**; *Kenneth Rosen, Massachusetts Institute of Technology:* **854.**

A Non-Unique Cryptarithm

859. [March, 1973] *Proposed by B. Suer and H. Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm $THREE + NINE = EIGHT + FOUR$.

I. *Solution by Harry L. Nelson, Livermore, California.*

There are 12 solutions in decimal base. They are:

THREE	+	*NINE*	=	*EIGHT*	+	*FOUR*
30122	+	4542	=	25703	+	8961
29433	+	7073	=	30692	+	5814
40233	+	5653	=	36104	+	9782
59766	+	4346	=	63295	+	0817
70566	+	2926	=	69107	+	4385
69877	+	5457	=	74096	+	1238

In each of these one can interchange the values of G and O to obtain another solution yielding 12 in all.

If one were to add the condition that "*THREE* is a prime" only the pair 69877 + 5457 = 74096 + 1238 = 74296 + 1038 would qualify; and if in addition we ask that "FOUR not be divisible by 3" the solution would be unique (base 10).

II. *Solution by John Tabor and John Beidler (jointly), University of Scranton, Pennsylvania.*

Solutions to additive cryptarithms are now trivia with the TABOR-AUTOMATIC CRYPTARITHM SOLVER. This program will accept any cryptarithm involving several additions and one equal sign and solve it in any base.

The program was written as a term project in a course on DATA STRUCTURES. The cryptarithm

$$THREE + NINE = FOUR + EIGHT$$

proved uninteresting in that it has 10 solutions. The replacements to obtain these solutions are:

E	*T*	*R*	*N*	*H*	*U*	*F*	*G*	*O*	*F*
2	3	1	4	0	6	5	7	9	8
2	3	1	4	0	6	5	9	7	8
3	2	4	7	9	1	0	6	8	5
3	2	4	7	9	1	0	8	6	5
3	4	2	5	0	8	6	1	7	9
3	4	2	5	0	8	6	7	1	9
6	7	5	2	0	8	9	1	3	4
6	7	5	2	0	8	9	3	1	4
7	6	8	5	9	3	4	0	2	1
7	6	8	5	9	3	4	2	0	1

Total CPU time was 20 seconds on an XDS Sigma 5. The program is in FORTRAN.

Also solved by Merrill Barnebey, University of Wisconsin at La Crosse; Harold Biller, Brooklyn, New York; Dorothy Brunet, Sherman Oaks, California; Robert Copus, Rose Hulman Institute of Technology; H. Marlon Hewit, Reedley High School, California; J. A. H. Hunter, Toronto, Canada; Janice A. McGoldrick, Cranston High School, Rhode Island; Sam Newman, Atlantic City, New Jersey; Erwin Schmidt, Washington, D. C.; S. O. Shachter, Philadelphia, Pennsylvania; Mary F. Turner, Glen Allen, Virginia; C. S. Venkataraman, Trichur, India; and the proposers.

Another Triangular Inequality

860. [March, 1973] *Proposed by Leon Bankoff, Los Angeles, California.*

In any triangle ABC show that

$$\sin A/2 + \sin B/2 + \sin C/2 \geqq \cos A + \cos B + \cos C.$$

Solution by E. P. Starke, Plainfield, New Jersey.

We start with

$$\tfrac{1}{2}(\cos A + \cos B) = \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) = \sin\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B).$$

Adding this and analogous formulas for other pairs of angles, we have

$$\cos A + \cos B + \cos C = \sin\frac{A}{2}\cos\frac{B-C}{2} + \sin\frac{B}{2}\cos\frac{A-C}{2} + \sin\frac{C}{2}\cos\frac{A-B}{2}.$$

Since each cosine is $\leqq 1$, the desired inequality follows immediately, with equality if and only if the triangle is equilateral.

(1) Note that the proposed inequality could be expressed as

$$\Sigma\left(4\sin\frac{A}{2} + 1\right)^2 \geqq 27.$$

(2) Compare Problem E1272, American Mathematical Monthly, 1960, p. 693, where it is proved that

$$\Sigma\sin\frac{A}{2} \leqq \left\{\frac{3}{2} + \frac{1}{2}\,(\Sigma\cos A)\right\}^{\frac{1}{2}}.$$

(3) Note further that, starting with

$$\frac{1}{2}(\sin A + \sin B) = \sin\frac{A+B}{2}\cos\frac{A-B}{2} = \cos\frac{C}{2}\cos\frac{A-B}{2} \leqq \cos\frac{C}{2}$$

the same method proves

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geqq \sin A + \sin B + \sin C.$$

(4) If now this inequality and the proposed inequality are multiplied member by member, the result is easily reduced to

$$2\sum \cos\frac{A}{2} \geqq \sum \sin A + \sum \sin 2A.$$

Also solved by Ralph Garfield, The College of Insurance, New York; Leonard Goldstone, Watervliet, New York; M. G. Greening, University of New South Wales, Australia; Graham Lord, Temple University; Murray S. Klamkin, Ford Motor Company, Michigan; Robert S. Stacy, Manzano High School, Albuquerque, New Mexico; Phil Tracy, Liverpool, New York; C. S. Venkataraman, Trichur, India; Atila Yanik, Illinois Institute of Technology; and the proposer.

Integral Zeroes

861. [March, 1973] *Proposed by Erwin Just, Bronx Community College.*

Find all integral values of m for which the function, f, defined by $f(x) = x^3 - mx^2 + mx - (m^2 + 1)$ has an integral zero.

Solution by William F. Fox, Moberly Area Junior College, Missouri.

Suppose p is an integer such that

$$p^3 - mp^2 + mp - (m^2 + 1) = 0.$$

It follows that

$$(p^2 + m)(p - m) = 1.$$

Since p and m are integers, then

(1) $$p^2 + m = p - m = -1$$

or

(2) $$p^2 + m = p - m = 1.$$

If (1) then $m = p + 1$, and so $p^2 + p + 1 = -1$ or $p^2 + p + 2 = 0$ which has no real solution. If (2) then $m = p - 1$, and so $p^2 + p - 1 = 1$ and so

$$p^2 + p - 2 = 0$$

which has the solutions $p = -2$, $p = 1$.

Hence, $m = -3$ and $m = 0$ are the integral values of m for which f has an integral zero.

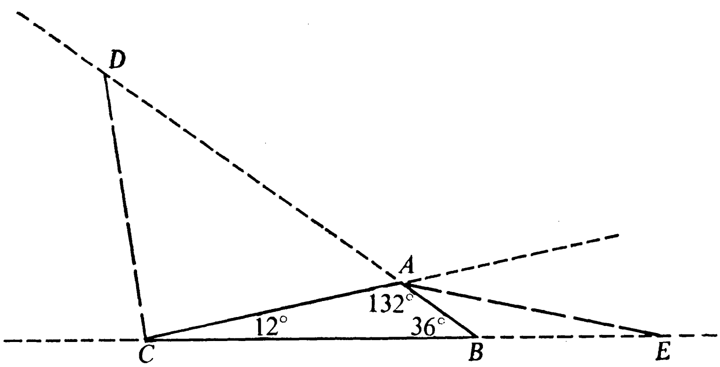
Also solved by Bernard August, Glassboro State College, New Jersey; Merle J. Biggin, Washington, D. C.; Joseph B. Browne, Oklahoma State University; Ragnar Dybvik, Tingvoll, Norway; Stanley Fox, City College of New York; Richard A. Gibbs and Harold Stocker, Fort Lewis College, Colorado (jointly); Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, *Australia; Richard Groeneveld, Iowa State University; Vaclav Konecny, Hawkins, Texas; Henry S. Lieberman, John Hancock Mutual Life, Massachusetts; V. Linis, University of Ottawa; Graham Lord, Temple University; John J. Moore, Niagara University, New York; C. F. Pinzka, University of Cincinnati; Paul Smith, University of Victoria; Robert S. Stacy, Manzano High School, Albuquerque, New Mexico; E. P. Starke, Plainfield, New Jersey; Phil Tracy, Liverpool, New York; Edward T. H. Wang, University of Waterloo, Canada; Kenneth M. Wilke, Topeka, Kansas; William T. Wood and Dale Woods, Northeast Missouri State College (jointly); K. L. Yocom, North Dakota State University; and the proposer.*

Not Necessarily Isosceles

862. [March, 1973] *Proposed by K. R. S. Sastry, Makele, Ethiopia.*

It is well known that the bisectors of equal angles of an isosceles triangle are equal. Conversely, the Steiner-Lehmus theorem states that if two internal bisectors of a triangle are equal, the triangle is isosceles. Show that if two external bisectors of a triangle are equal, the triangle need not be isosceles.

I. *Solution by Charles W. Trigg, San Diego, California.*



Consider the scalene triangle ABC in which the interior angles $A = 132°$, $B = 36°$, and $c = 12°$. The external bisector at C meets BA extended at D. The external bisector at A meets CB extended at E.

$$\angle ACD = (180° - 12°)/2 = 84°.$$

In triangle BCD, $\angle CDA = 180° - 12° - 84° - 36° = 48°$.

$$\angle CAD = 180° - 132° = 48° = \angle CDA, \text{ so } CD = CA.$$

$$\angle BAE = (180° - 132°)/2 = 24°.$$

In triangle CAE, $\angle CEA = 180° - 24° - 132° - 12° = 12° = \angle ACE$, so $AE = CA = CD$. That is, the two external bisectors are equal, and the scalene triangle is pseudo-isosceles.

More generally, the square of an external bisector, T_A, of angle A of a triangle is equal to $bc[a^2/(b-c)^2 - 1]$. Thus, if $T_C = T_A$, then $ab[c^2/(b-a)^2 - 1] = bc[a^2/(b-c)^2 - 1]$. This expression simplifies to

$$b(a-b+c)(c-a)[b^3-(a+c)b^2+3acb-ac(a+c)]=0.$$

The first two factors cannot be zero, so either $(c-a)=0$ and the triangle is isosceles, or

$$(1) \qquad b^3-(a+c)b^2+3acb-ac(a+c)=0.$$

This equation in b has one real and two complex roots. The real root is

$$b = M + \sqrt[3]{M^3+\sqrt{N}} + \sqrt[3]{M^3-\sqrt{N}},$$

where $M=(a+c)/3$ and the always positive $N = ac[27a^2c^2-9ac(a+c)^2 + (a+c)^4]/27$. When $a=c$, the corresponding triangle is equilateral. When $a \neq c$, the triangle is pseudo-isosceles.

After multiplication by $a+b+c=2s$, the equation (1) can be manipulated into the form

$$[(s-b)/b]^2 = [(s-a)/a][(s-c)/c].$$

This expression, in turn, is equivalent to

$$\sin^2\tfrac{1}{2}B = \sin\tfrac{1}{2}A\sin\tfrac{1}{2}C,$$

of which the triangle opening this discussion is a special case.

This problem has been frequently discussed previously, for example, in American Mathematical Monthly, 24(1917), 344; 40(1933), 423; 45(1938), 480; National Mathematics Magazine, 14(1939), 51; School Science and Mathematics, 31(1931), 465; 39(1939), 563, 732–735; 40(1940), 464–468.

II. *Solution by Lawrence A. Ringenberg, Eastern Illinois University.*

In an xy-coordinate plane consider the scalene right triangle ABC with $A=(0,0)$, $B=(0,b)$, $C=(1,0)$, where $b>1$. Let A' and B' be the points on rays $\overrightarrow{BC}$ and $\overrightarrow{CA}$ such that $\overline{AA'}$ and $\overline{BB'}$ are external angle bisectors at A and B, respectively. Then

$$A' = \left(\frac{b}{b-1}, \frac{-b}{b-1}\right),\ B' = (b - b\sqrt{b^2+1}, 0),\ AA' = \frac{b\sqrt{2}}{b-1},$$

and

$$BB' = (2b^4+2b^3\sqrt{b^2+1}+2b^2)^{\frac{1}{2}}.$$

Since AA' and BB' are continuous functions of b, AA' decreasing from ∞ to $\sqrt{2}$ and BB' increasing from $\sqrt{4+2\sqrt{2}}$ to ∞ as b increases from 1 to ∞, it follows that there is exactly one value of b such that $AA'=BB'$. Therefore there is a scalene triangle with two external angle bisectors of the same length. Using a desk calculator we find that $AA' \approx BB' \approx 4.734$ if $b=1.426$.

Also solved by Timothy Gallagher, Rose Hulman Institute of Technology; Michael Goldberg, Washington, D. C.; Ranee Gupta, University-Liggett School, Michigan; Vaclav Konecny, Hawkins, Texas; Joseph D. E. Konhauser, Macalester College, Minnesota; C. C. Oursler, Southern Illinois University; E. P. Starke, Plainfield, New Jersey; Phil Tracy, Liverpool, New York; C. S. Venkataraman, Trichur, India; Atila Yanik, Illinois Institute of Technology; and the proposer.

Property of a Hadamard Matrix

863. [March, 1973] *Proposed by K. W. Schmidt, University of Manitoba, Canada.*

The number of (-1)'s of an n-order Hadamard matrix is bounded by $n[n \pm (\sqrt{2n-1}-1)]/2$.

Solution by Michael Goldberg, Washington, D.C.

The Hadamard matrix, in normal form, has the minimum number of negative entries, namely, $n(n-1)/2$. (See *Combinatorial Mathematics*, H. J. Ryser, 1963, p. 104.) If all the positive entries in the first row or column are changed to negative entries, then the maximum number of negative entries are obtained, namely, $n(n+1)/2$.

For $n > 3$, $n(n-1)/2 > n[n-(\sqrt{2n-1}-1)]/2$, and $n(n+1)/2 < n[n+(\sqrt{2n-1}-1)]/2$.

Also solved by Edward T. H. Wang, University of Waterloo, Canada; and the proposer.

Prime and Composite Sums

864. [March, 1973] *Proposed by Charles W. Trigg, San Diego, California.*

In the square array

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 6 & 9 \end{matrix}$$

all but two of the twelve adjacent digit pairs, taken horizontally and vertically, have prime sums.

(a) Show that it is impossible to rearrange the digits so that every pair of adjacent digits has a prime sum.

(b) Show that the digits can be rearranged so that each of the twelve sums of adjacent digits is composite, with nine of the composite sums being distinct.

I. *Solution by M. T. Bird, California State University, San Jose.*

Pairs of the numbers from one to nine have fifteen distinct sums of which six are odd primes and nine are composite.

(a) If each of the twelve sums arising from pairs of adjacent digits of the array

$$\begin{matrix} a & p & b \\ q & c & r \\ d & s & e \end{matrix}$$

are to be prime then even and odd digits must alternate in both rows and columns. We conclude that $abcde$ is an arrangement of the odd digits and $pqrs$ is an arrangement of the even digits. Essentially three possibilities exist: $p = 2$ and $s = 4, 6$, or 8.

Suppose $pqrs = 2684$. Then $d = 7$ because 7 cannot be adjacent to 2 or 8, $a = 1$ because 1 cannot be adjacent to 8, $b = 5$ because 5 cannot be adjacent to 4, and $e = 3$ because 3 cannot be adjacent to 6. But we have $c = 9$ and 9 is adjacent to 6 which is prohibited.

Suppose $pqrs = 2486$. Then $d = 7$ because 7 cannot be adjacent to 2 or 8, $a = 1$ because 1 cannot be adjacent to 8, $b = 3$ because 3 cannot be adjacent to 6, and $e = 5$ because 5 cannot be adjacent to 4. But we have $c = 9$ and 9 is adjacent to 6 which is prohibited.

Suppose $pqrs = 2468$. There is no place for 7 because 7 must not be adjacent to 2 or 8.

We have exhausted possibilities and proved (a).

(b) Consider the array

6	2	4
9	7	8
5	3	1

This one example completes the proof.

II. *Solution by the proposer.*

The sum of two odd digits is divisible by 2. To have all sums composite, each odd digit must be adjacent to an odd digit, or to an even digit such that the sum of the pair is a multiple of 3. Consequently three odd digits must fall along a side with the additional two completing a 2-by-2 array. Moreover, the odd central digit must form multiples of 3 with two different even digits. This establishes the positions of 7, 8, 2, and 1. The other digits can be distributed in the remaining positions to form the following six different arrays with composite sums:

3	5	4	9	5	4	5	3	6	9	3	6	3	9	6	5	9	6
9	7	2	3	7	2	9	7	2	5	7	2	5	7	2	3	7	2
1	8	6	1	8	6	1	8	4	1	8	4	1	8	4	1	8	4

In the given order of the arrays, their twelve pair-sums (multiple occurrence of a particular sum follows it in parentheses, and the sum of the twelve pair-sums is in brackets at the end of each sum set) are:

6, 8(2), 9(3), 10, 12(2), 14, 15, 16, [128];

4, 6, 8, 9(3), 10, 12(2), 14(2), 15, 16, [122];

6, 8(2), 9(3), 10(2), 12, 14, 15, 16, [126];

6(2), 8, 9(3), 10, 12(3), 14, 15, [122];

6(2), 8(2), 9(2), 12(3), 15(2), 16, [128]; and

4, 6, 8(2), 9(2), 10, 12, 14, 15(2), 16, [126].

The six sum sets are distinct. The number of distinct sums ranges from six in the fifth array to nine in the last array.

None of the arrays is antimagic.

Also solved by Donald Braffitt, Armstrong State College, Georgia; Eliot W. Collins, New Paltz, New York; Clayton W. Dodge, University of Maine at Orono; Wanda Dunbar, Whitman College, Washington; Ralph E. Edwards, Baltimore Life Insurance Co., Maryland; William F. Fox, Moberly Junior College, Missouri; Timothy Gallagher, Rose Hulman Institute of Technology; M. G. Greening, University of New South Wales, Australia; Ranee Gupta, University-Liggett School, Michigan (two solutions); Howard Lyn Hiller, Cornell University; Vaclav Konecny, Hawkins, Texas; M. S. Krishnamorthy, Kanpur, India; N. J. Kuenzi and Bob Prielipp, University of Wisconsin at Oskosh (jointly); Harry L. Nelson, Livermore, California; S. O. Shachter, Philadelphia, Pennsylvania; Paul Smith, University of Victoria; Robert S. Stacy, Manzano High School, Albuquerque, New Mexico; Jim Tattersall, Attleboro, Massachusetts; Phil Tracy, Liverpool, New York; Wolf R. Umbach, Rottdorf, Germany; Edward T. H. Wang, University of Waterloo, Canada; Kenneth M. Wilke, Topeka, Kansas; J. W. Wilson, Athens, Georgia; and the proposer.

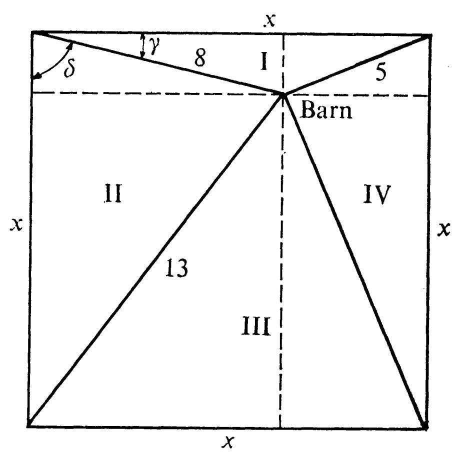
Locating the Barn

865. [March, 1973] *Proposed by A. Polter Geist, Miskatonic University, Arkham, Massachusetts.*

A square tract of land is bounded by four roads. A barn is located on the tract exactly 13 miles from the southwest corner, 8 miles from the northwest corner, and 5 miles from the northeast corner. How far is the barn from the nearest road?

I. *Solution by Joseph V. Michalowicz, Catholic University of America.*

The tract is depicted by the following diagram.



The distance from the barn to the nearest road is the smallest of the altitudes (shown by dashed lines) of the 4 triangles I, II, III, IV. The Law of Cosines applied to triangles I and II gives the equations:

$$25 = 64 + x^2 - 16x\cos\gamma$$

$$169 = 64 + x^2 - 16x\cos\delta.$$

Since $\gamma + \delta = 90°$ and so $\cos\delta = \sin\gamma$, these equations can be rearranged to give

$$\text{(1)} \qquad \cos\gamma = \frac{x^2 + 39}{16x}$$

$$\text{(2)} \qquad \sin\gamma = \frac{x^2 - 105}{16x}.$$

Substitution of these equalities into the identity $\cos^2\gamma + \sin^2\gamma = 1$ leads to the following fourth order equation:

$$x^4 - 194x^2 + 6273 = 0.$$

The quadratic formula gives the solutions $x^2 = 153, 41$. The second of these solutions is discarded since it would make $\sin\gamma$ negative in equation (2), so $x^2 = 153$ and $x \approx 12.37$.

Either (1) or (2) then gives $\gamma \approx 14°2'$ and $\delta \approx 75°58'$. Therefore, the altitude alt (I) of triangle I is given by

$$\text{alt (I)} = 8\sin\gamma \approx 1.94.$$

Also

$$\text{alt (II)} = 8\sin\delta \approx 7.76$$

$$\text{alt (III)} = x - \text{alt (I)} \approx 10.43$$

$$\text{alt (IV)} = x - \text{alt (II)} \approx 4.61.$$

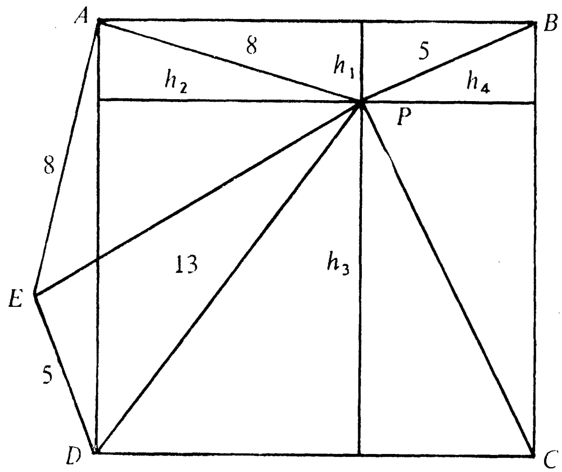
So the required distance is 1.94 miles.

II. *Solution by Mannis Charosh, Brooklyn, New York.*

By a well-known theorem in elementary geometry: $PA^2 + PC^2 = PB^2 + PD^2$. Therefore $PC = \sqrt{130}$.

We can now show that P is nearest to the road AB. Let the 4 distances be represented by h_1, h_2, h_3, h_4, as in the diagram. In ΔABP and ΔADP, $AB = AD$, $AP = AP$, and $PB < PD$. Hence $\angle PAB < \angle PAD$; so that $h_1 < h_2$. Similarly it can be shown that $h_2 < h_3$, and $h_1 < h_4$. Thus h_1 is the distance we seek.

To find h_1, we first find the length of the side of the square. Rotate ΔABP so that AB coincides with AD, and E falls outside of the square. We find in turn: (1) $\angle BAP = \angle DAE$; (2) $\angle PAE = 90°$; (3) $\angle AEP = 45°$; (4) $EP = 8\sqrt{2}$; (5) $\cos\angle PED = -1/5\sqrt{2}$; $\cos\angle AED = -4/5$; $AD = 3\sqrt{17}$. It is then easy, given the sides of ΔABP, to calculate that $h_1 = 8/\sqrt{17}$.



In confirmation, we find that $h_2 = 32/\sqrt{17}$; $h_3 = 43/\sqrt{17}$, and $h_4 = 19/\sqrt{17}$.

III. *Charles W. Trigg, San Diego, California, gave the following references.*

Some references to previous considerations of this configuration (with different distances) are:

American Mathematical Monthly, 3(1896)155; 35(1928)94, 383; 47(1940)396; 50(1943)392;

National Mathematics Magazine, 12(1937)141; 16(1941)106; 17(1942)39;

School Science and Mathematics, 15(1915)632; 32(1932)788; 46(1946)89, 783; 50(1950)324; 59(1959)500.

Also solved by Mangho Ahuja and Jack Gosnell, Southeast Missouri State College (jointly); Bernard August, Glassboro State College, New Jersey; Merrill Barnebey, Holmen, Wisconsin; Merle J. Biggin, Washington, D. C.; Wray G. Brady, Slippery Rock State College, Pennsylvania; Martin J. Brown, University of Kentucky; Joseph B. Browne, Oklahoma State University; Brenda Bryant, Southeast Missouri State University; Eliot W. Collins, New Paltz, New York; Steven R. Conrad, Benjamin N. Cardozo High School, New York; Clayton W. Dodge, University of Maine at Orono; Ralph E. Edwards, Baltimore Life Insurance Company, Maryland; Stanley Fox, City College of New York; William F. Fox, Moberly, Missouri; Ralph Garfield, The College of Insurance, New York; Jack Giammerse, Jr., Baton Rouge, Louisiana; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Ranee Gupta, University-Liggett School, Michigan; Robert M. Hashway, West Warwick, Rhode Island; John Heath, University of Maine; John M. Howell, Littlerock, California; J. A. H. Hunter, Toronto, Canada; Frank Kocher, Pennsylvania State University; Randy Kolod, College of St. Rose, New York; Vaclav Konecny, Hawkins, Texas; Lew Kowarski, Morgan State College, Maryland; Sidney Kravitz, Dover, New Jersey; Henry S. Lieberman, Hancock

Mutual Life Insurance Co., Massachusetts; Hubert J. Ludwig, Ball State University, Indiana; Thomas J. Miles, Central Michigan University; David Montgomery, Rose-Hulman Institute of Technology; Harry C. Nelson, Livermore, California; C. Stanley Ogilvy, Hamilton College, New York; John E. Prussing, University of Illinois-Urbana; Lawrence A. Ringenberg, Eastern Illinois University; M. Rodeen, San Mateo, California; Eugene S. Santos, Youngstown State University, Ohio; Erwin Schmidt, Washington, D. C.; S. O. Shachter, Philadelphia, Pennsylvania; Paul Smith, University of Victoria; Robert C. Stacy, Manzano High School, Albuquerque, New Mexico; E. P. Starke, Plainfield, New Jersey; Jim Tattersall, Attleboro, Massachusetts; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; Wolf R. Umbach, Rottdorf, Germany; Edward T. H. Wang, University of Waterloo, Canada; Kenneth M. Wilke, Topeka, Kansas; J. W. Wilson, Athens, Georgia; Atila Yanik, Illinois Institute of Technology; and the proposer.

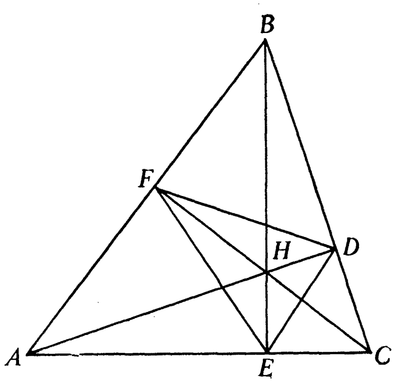
Comment on Problem 837

837. [May, 1972; March, 1973] *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Prove that the altitudes of any triangle bisect the angles of another triangle whose vertices are the feet of the altitudes of the first triangle.

Comment by Charles W. Trigg.

It is not necessary to resort to trigonometry or to auxiliary lines for a proof.



In the triangle ABC, the altitudes AD, BE, and CF intersect at the orthocenter H. Right triangles ABE and ACF have a common acute angle, so $\angle ABE = \angle ACF$.

Quadrilaterals $FHDB$ and $EHDC$ are inscriptible, so $\angle FBH = \angle FDH$ and $\angle HDE = \angle HCE$. Hence, $\angle FDH = \angle FBH = \angle ABE = \angle ACF = \angle HCE = \angle HDE$ and AD bisects $\angle FDE$. Similar arguments apply to the altitudes BE and CF.

Therefore, the altitudes of a triangle are the internal bisectors of the angles of its orthic triangle.

Other synthetic proofs of this well-known proposition appear on Pages 85 and 86 of N. A. Court's *College Geometry*, Johnson Publishing Co. (1925).

Comment on Q566

Q566. [March, 1973] The arithmetic mean of twin primes 5 and 7 is a perfect number 6. Are there other twin primes with a perfect mean?

[*Submitted by Charles W. Trigg*]

Comment by Joseph Silverman, White Plains, New York.

A much simpler proof is available which does not use the general form for a perfect number and yields a more general result. Except for 3, 5 and 5, 7, the arithmetic mean of all twin primes is of the form $6k$ with $k > 1$ and k an integer. $6k$ is divisible by the distinct numbers $3k$, $2k$, k, 1. The sum of the divisors is $6k + 1$. Therefore $6k$ is abundant.

ANSWERS

A583. Define $g(x) = (x^a - 1)f(x)$. On $[0,1]$ g satisfies the hypothesis of Rolle's Theorem. Therefore, there exists $c \in (0,1)$ such that $0 = g'(c) = ac^{a-1}f(c) + (c^a - 1)f'(c)$, which yields $f'(c) = ac^{a-1}f(c)/(1 - c^a)$.

A584. The inequality can be rewritten as $4\,\Delta \geqq 8 \sin A \sin B \sin C$ or $1 \geqq R$, where Δ and R are area and circumradius, respectively, of a triangle of sides $2 \sin A$, $2 \sin B$ and $2 \sin C$. If we inscribe the triangle in a unit sphere, we obtain a corresponding spherical triangle of sides $2A$, $2B$, $2C$ whose circumradius R is then $\leqq 1$. The equality occurs when the spherical triangle corresponds to a great circle.

A585. Lucky Larry used the decimal system and wrote $2^5 9^2 = 2592$. An analysis of the relation $a^b c^a = ar^3 + br^2 + cr + a$ for four digits a, b, c each less than a positive integer r shows that the relation is true only for $r = 10$, in which case, $a = 2$, $b = 5$ and $c = 9$.

A586. Both sides equal

$$\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{1}{(m+n+1)\binom{m+n}{n}}$$

which comes from the fact that

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha+1}(1-x)^{\beta+1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\beta+1}(1-x)^{\alpha+1}dx = 1.$$

A587. By Fermat's Theorem $p^4 \equiv 1 \pmod 5$ whence $(p^2 - 1)(p^2 + 1) \equiv 0 \pmod 5$. Thus since p is odd, both $p^2 - 1$ and $p^2 + 1$ are even, and neither $10 \mid p^2 - 1$ nor $10 \mid p^2 + 1$.

(Quickies on page 48)

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